



RLS Wiener Filter and Fixed-Point Smoother with Randomly Delayed or Uncertain Observations in Linear Discrete-Time Stochastic Systems

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Abstract

This paper designs the recursive least-squares (RLS) Wiener fixed-point smoother and filter from randomly delayed observed values by multiple sampling times or uncertain observations in linear discrete-time stochastic systems. The observed value is generated in terms of the delayed observed values or uncertain observed values. In the case of the observed value with delay or without delay, their probabilities are assigned. Here, each observation includes signal plus white observation noise. Related to the uncertain observed value with delay or without delay, the probability that the observation consists of only observation noise is allocated, according to the time delayed or not delayed. It is assumed that the delay and uncertain measurements are characterized by the Bernoulli random variables. The RLS Wiener estimators use the following information. (1) The system matrix. (2) The observation matrix. (3) The variance of the state vector. (4) The probabilities concerned with the delayed observation and the uncertain observation. (5) The variance of white observation noise.

Keywords: Discrete-time stochastic system; RLS Wiener filter; RLS Wiener fixed-point smoother; randomly delayed observations; uncertain observations

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Introduction

Data in a network of control system are usually transmitted with delay and packet dropout from a sensor to a controller and also from a controller to an actuator [1]. In linear discrete-time stochastic systems, the optimal filter, predictor and smoother are proposed, based on the innovation approach, from observations with randomly delayed and packet dropouts [2], [3]. Also, with observations multiply and randomly delayed, the recursive least-squares (RLS) Wiener fixed-point smoother and filter are proposed [4]. Concerned with the uncertain observation, the RLS estimation method is presented, given the probability that the signal exists in the observation characterized by the independent Bernoulli random variable [5]. Estimation technique in [5] is extended to the case where the random variable, related to the uncertainty, in the observation equation has the correlation property [6]. In addition to the probability that the signal exists in the observation, the conditional probability is given for the existence of the signal in the observation.

In [7], the RLS Wiener fixed-point smoother and filter are proposed for the uncertain observation. The probability that the signal exists in the observation and the conditional probability are used in the estimators. In wireless sensor network system with multiple packet losses, related to the delayed observations, the optimal filter is proposed in linear discrete-time stochastic systems [8]. The technique is also extended to the design of the filter in nonlinear discrete-time stochastic systems. In [9], the quadratic filter is proposed for the nonlinear discrete-time stochastic systems. The nonlinear state equation and the uncertain nonlinear observation equation, characterized by the Bernoulli random variables, are given.

Hitherto, it seems that there have been no studies on the RLS Wiener estimation problem considering the both cases of the multiply delayed or uncertain observations. From this viewpoint, this paper designs the RLS Wiener fixed-point smoother and filter from randomly delayed observed values by multiple sampling times or uncertain observations in linear discrete-time stochastic systems. The observed value is generated in terms of the delayed observed values or uncertain observed values. In the case of the observed value with delay or without delay, their probabilities are assigned. Here, each observation includes signal plus white observation noise. Related to the uncertain observed value with delay or without delay, the probability that the observation consists of only observation noise is allocated, according to the time delayed or not delayed. It is assumed that the delay and uncertain measurements are characterized by the Bernoulli random variables. The RLS Wiener estimators use the following information. (1) The system matrix. (2) The observation matrix. (3) The variance of the state vector. (4) The probabilities concerned with the delayed observation and the uncertain observation. (5) The variance of white observation noise.

A numerical simulation example, in section 5, shows the estimation characteristics of the current filter and the fixed-point smoother with the multiply delayed or uncertain observations.

Least-squares fixed-point smoothing problem

Let an m -dimensional observation equation be given by

$$\begin{aligned} y(k) &= \gamma_{01}(k)\bar{y}(k) + \gamma_{11}(k)\bar{y}(k-1) + \cdots + \gamma_{\bar{N}1}(k)\bar{y}(k-\bar{N}) \\ &+ \gamma_{00}(k)v(k) + \gamma_{10}(k)v(k-1) + \cdots + \gamma_{\bar{N}0}(k)v(k-\bar{N}), \\ \bar{y}(k) &= z(k) + v(k), \quad z(k) = Hx(k), \\ E[\gamma_{01}(k)] &= p_{01}(k), \quad E[\gamma_{11}(k)] = p_{11}(k), \quad E[\gamma_{21}(k)] = p_{21}(k), \dots, \quad E[\gamma_{\bar{N}1}(k)] = p_{\bar{N}1}(k), \\ E[\gamma_{00}(k)] &= p_{00}(k), \quad E[\gamma_{10}(k)] = p_{10}(k), \quad E[\gamma_{20}(k)] = p_{20}(k), \dots, \quad E[\gamma_{\bar{N}0}(k)] = p_{\bar{N}0}(k), \end{aligned} \quad (1)$$

in linear discrete-time stochastic systems. It is assumed that the observation at each time $k > 1$ can be either delayed by sampling periods, j , $1 \leq j \leq \bar{N}$, with known probabilities or consist of only delayed observation noise data without including signal data. $\{\gamma_{ij}(k), 0 \leq i \leq \bar{N}, j = 0, 1; \quad k > 1\}$ denote a sequence of Bernoulli random variables (binary switching sequence taking the values 0 or 1 with



$P[\gamma_{ij}(k) = 1] = p_{ij}(k), 0 \leq i \leq \bar{N}, j = 0, 1$. Usually, in applications of communication networks, $\{\gamma_j(k); k \geq 1\}$ represents the random delay from sensor to [4]. Here, $z(k)$ is a signal vector, H is an m by n observation matrix, $x(k)$ is a state vector and $v(k)$ is white observation noise. It is assumed that the signal and the observation noise are mutually independent and have the mean of zero respectively. Let the auto-covariance function of $v(k)$ be given by

$$E[v(k)v^T(s)] = R\delta_K(k-s), R > 0. \quad (2)$$

Here, $\delta_K(\cdot)$ denotes the Kronecker delta function.

By denoting

$$\begin{aligned} \bar{\gamma}_1(k) &= [\gamma_{01}(k)I_{m \times m} \quad \gamma_{11}(k)I_{m \times m} \quad \gamma_{21}(k)I_{m \times m} \quad \cdots \quad \gamma_{\bar{N}1}(k)I_{m \times m}], \\ \bar{\gamma}_0(k) &= [\gamma_{00}(k)I_{m \times m} \quad \gamma_{10}(k)I_{m \times m} \quad \gamma_{20}(k)I_{m \times m} \quad \cdots \quad \gamma_{\bar{N}0}(k)I_{m \times m}], \\ \tilde{y}(k) &= [\bar{y}(k) \quad \bar{y}(k-1) \quad \cdots \quad \bar{y}(k-\bar{N})]^T, \\ \bar{v}(k) &= [v(k) \quad v(k-1) \quad \cdots \quad v(k-\bar{N})]^T, \end{aligned} \quad (3)$$

from (1), we obtain

$$y(k) = \bar{\gamma}_1(k)\tilde{y}(k) + \bar{\gamma}_0(k)\bar{v}(k). \quad (4)$$

Let $E_\gamma[\cdot]$ denote the expectation with respect to the random variables $\{\gamma(k), k \geq 1\}$.

The Bernoulli random variables satisfy $E_\gamma[\gamma_{ij}(k)] = p_{ij}(k)I_{m \times m}$, $E_\gamma[\gamma_{ij}^2(k)] = p_{ij}(k)I_{m \times m}$, $0 \leq i \leq \bar{N}, j = 0, 1$.

From (3), it is seen that the auto-covariance function of $\bar{v}(k)$ is given by

$$K_{\bar{v}}(k, s) = \begin{cases} \bar{C}(k)\bar{D}^T(s), 0 \leq s \leq k, \\ \bar{D}(k)\bar{C}^T(s), 0 \leq k \leq s, \end{cases} \quad (5)$$

$\bar{C}(k) = \Phi_{\bar{v}}^k, \bar{D}^T(s) = \Phi_{\bar{v}}^{-s}K_{\bar{v}}(s, s)$. Here, the transition matrix $\Phi_{\bar{v}}$ and the variance $K_{\bar{v}}(s, s)$ of $\bar{v}(k)$ are given by

$$\Phi_{\bar{v}} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ I_{m \times m} & 0 & \cdots & 0 & 0 \\ 0 & I_{m \times m} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_{m \times m} & 0 \end{bmatrix}, K_{\bar{v}}(s, s) = \begin{bmatrix} R & 0 & \cdots & 0 & 0 \\ 0 & R & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & R & 0 \\ 0 & 0 & \cdots & 0 & R \end{bmatrix}. \quad (6)$$

By denoting

$$\bar{H} = \begin{bmatrix} H & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & H & 0 \\ 0 & \cdots & 0 & H \end{bmatrix}, \bar{x}(k) = \begin{bmatrix} x(k) \\ x(k-1) \\ \vdots \\ x(k-\bar{N}) \end{bmatrix}, \quad (7)$$

from (1) and (3), the observation equation (4) is also written as

$$y(k) = \bar{\gamma}_1(k)\bar{H}\bar{x}(k) + \bar{\gamma}_1(k)\bar{v}(k) + \bar{\gamma}_0(k)\bar{v}(k), \quad (8)$$



since

$$\bar{z}(k) = \begin{bmatrix} z(k) \\ z(k-1) \\ \vdots \\ z(k-\bar{N}) \end{bmatrix} = \begin{bmatrix} Hx(k) \\ Hx(k-1) \\ \vdots \\ Hx(k-\bar{N}) \end{bmatrix} = \bar{H}\bar{x}(k). \quad (9)$$

Let $K_x(k, s) = K_x(k-s)$ represent the auto-covariance function of the state vector $x(k)$ in wide-sense stationary stochastic systems [16], and let $K_x(k, s)$ be expressed in the form of

$$K_x(k, s) = \begin{cases} A(k)B^T(s), & 0 \leq s \leq k, \\ B(s)A^T(k), & 0 \leq k \leq s, \end{cases} \quad (10)$$

$A(k) = \Phi^k$, $B^T(s) = \Phi^{-s}K_x(s, s)$. Here, Φ is the transition matrix of $x(k)$.

Let the state-space model for $x(k)$ be described as

$$x(k+1) = \Phi x(k) + Bw(k), \quad E[w(k)w^T(s)] = Q\delta_K(k-s), \quad (11)$$

where B is n by l input matrix and $w(k)$ is white input noise with the auto-covariance function of (10). Let $\bar{\Phi}$ represent the system matrix for $\bar{x}(k)$. From

$$\begin{bmatrix} x(k+1) \\ x(k) \\ \vdots \\ x(k-\bar{N}+2) \\ x(k-\bar{N}+1) \end{bmatrix} = \begin{bmatrix} \Phi & 0 & \cdots & 0 & 0 \\ I_{n \times n} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & I_{n \times n} & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ x(k-1) \\ \vdots \\ x(k-\bar{N}+1) \\ x(k-\bar{N}) \end{bmatrix} + \begin{bmatrix} w(k) \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad (12)$$

$\bar{\Phi}$ is given by

$$\bar{\Phi} = \begin{bmatrix} \Phi & 0 & \cdots & 0 & 0 \\ I_{n \times n} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & I_{n \times n} & 0 \end{bmatrix}. \quad (13)$$

Let $K_{\bar{x}}(k, s)$ represent the auto-covariance function of $\bar{x}(k)$. Then $K_{\bar{x}}(k, s)$ is expressed in the form of

$$K_{\bar{x}}(k, s) = \begin{cases} \bar{A}(k)\bar{B}^T(s), & 0 \leq s \leq k, \\ \bar{B}(k)\bar{A}^T(s), & 0 \leq k \leq s, \end{cases} \quad (14)$$

$\bar{A}(k) = \bar{\Phi}^k$, $\bar{B}^T(s) = \bar{\Phi}^{-s}K_{\bar{x}}(s, s)$. Here, $K_{\bar{x}}(s, s) = K_{\bar{x}}(0)$ is expressed as



$$K_{\bar{x}}(s, s) = E \left[\begin{bmatrix} x(s) \\ x(s-1) \\ \vdots \\ x(s-\bar{N}+1) \\ x(s-\bar{N}) \end{bmatrix} \begin{bmatrix} x^T(s) & x^T(s-1) & \cdots & x^T(s-\bar{N}+1) & x^T(s-\bar{N}) \end{bmatrix} \right] \quad (15)$$

$$= \begin{bmatrix} K_x(0) & \Phi K_x(0) & \cdots & \Phi^{\bar{N}-1} K_x(0) & \Phi^{\bar{N}} K_x(0) \\ K_x(0)\Phi^T & K_x(0) & \cdots & \Phi^{\bar{N}-2} K_x(0) & \Phi^{\bar{N}-1} K_x(0) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ K_x(0)(\Phi^T)^{\bar{N}-1} & K_x(0)(\Phi^T)^{\bar{N}-2} & \cdots & K_x(0) & \Phi K_x(0) \\ K_x(0)(\Phi^T)^{\bar{N}} & K_x(0)(\Phi^T)^{\bar{N}-1} & \cdots & K_x(0)\Phi^T & K_x(0) \end{bmatrix}.$$

Let the fixed-point smoothing estimate $\hat{\bar{x}}(k, L)$ of $\bar{x}(k)$ at the fixed point k be expressed by

$$\hat{\bar{x}}(k, L) = \sum_{i=1}^L h(k, i, L) y(i) \quad (16)$$

in terms of the observed values $\{y(i), 1 \leq i \leq L\}$. In (16), $h(k, i, L)$ is a time-varying impulse response function.

Let us consider the estimation problem, which minimizes the mean-square value (MSV)

$$J = E[\|\bar{x}(k) - \hat{\bar{x}}(k, L)\|^2] \quad (17)$$

of the fixed-point smoothing error. From an orthogonal projection lemma [16],

$$\bar{x}(k) - \sum_{i=1}^L h(k, i, L) y(i) \perp y(s), 1 \leq s \leq L, \quad (18)$$

the impulse response function satisfies the Wiener-Hopf equation

$$E[\bar{x}(k) y^T(s)] = \sum_{i=1}^L h(k, i, L) E[y(i) y^T(s)]. \quad (19)$$

Here ' \perp ' denotes the notation of the orthogonality.

By putting $\bar{p}_1(k) = [p_{01}(k)I_{m \times m} \quad p_{11}(k)I_{m \times m} \quad \cdots \quad p_{N-11}(k)I_{m \times m} \quad p_{N1}(k)I_{m \times m}]$, from (3) and (4), the left hand side of (19) is rewritten as

$$\begin{aligned} E[\bar{x}(k) y^T(s)] &= E[\bar{x}(k)(\bar{\gamma}_1(s)\bar{H}\bar{x}(s) + \bar{\gamma}_1(s)\bar{v}(s) + \bar{\gamma}_0(s)\bar{v}(s))^T] \\ &= E[\bar{x}(k)\bar{x}^T(s)]\bar{H}^T [p_{01}(s)I_{m \times m} \quad p_{11}(s)I_{m \times m} \quad \cdots \quad p_{N-11}(s)I_{m \times m} \quad p_{N1}(s)I_{m \times m}]^T \\ &= K_{\bar{x}}(k, s)\bar{H}^T \bar{p}_1^T(s). \end{aligned} \quad (20)$$

Also, from (2) and (8), $E[y(i) y^T(s)]$ is rewritten as

$$\begin{aligned} E[y(i) y^T(s)] &= E[(\bar{\gamma}_1(i)\bar{H}\bar{x}(i) + \bar{\gamma}_1(i)\bar{v}(i) + \bar{\gamma}_0(i)\bar{v}(i))(\bar{\gamma}_1(s)\bar{H}\bar{x}(s) + \bar{\gamma}_1(s)\bar{v}(s) + \bar{\gamma}_0(s)\bar{v}(s))^T] \\ &= E_{\gamma}[(\bar{\gamma}_1(i)\bar{H}K_{\bar{x}}(i, s)\bar{H}^T \bar{\gamma}_1^T(s)) + E_{\gamma}[\bar{\gamma}_2(i)K_{\bar{v}}(i, s)\bar{\gamma}_2^T(s)], \\ \bar{\gamma}_2(s) &= \bar{\gamma}_0(s) + \bar{\gamma}_1(s). \end{aligned} \quad (21)$$



Substituting (20) and (21) into (19), we obtain

$$K_{\bar{x}}(k, s)H^T \bar{p}_1^T(s) = \sum_{i=1}^L h(k, i, L) \{E_{\gamma}[\bar{\gamma}_1(i)\bar{H}K_{\bar{x}}(i, s)\bar{H}^T \bar{\gamma}_1^T(s)] + E_{\gamma}[\bar{\gamma}_2(i)K_{\bar{v}}(i, s)\bar{\gamma}_2^T(s)]\}. \quad (22)$$

From the stochastic property of $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$, (22) becomes

$$\begin{aligned} K_{\bar{x}}(k, s)H^T \bar{p}_1^T(s) &= h(k, s, L) \{E_{\gamma}[\bar{\gamma}_1(s)\bar{H}K_{\bar{x}}(s, s)\bar{H}^T \bar{\gamma}_1^T(s)] + E_{\gamma}[\bar{\gamma}_2(s)K_{\bar{v}}(s, s)\bar{\gamma}_2^T(s)] \\ &\quad - E_{\gamma}[\bar{\gamma}_1(s)]\bar{H}K_{\bar{x}}(s, s)\bar{H}^T E_{\gamma}[\bar{\gamma}_1^T(s)] - E_{\gamma}[\bar{\gamma}_2(s)]K_{\bar{v}}(s, s)E_{\gamma}[\bar{\gamma}_2^T(s)]\} \\ &\quad + \sum_{i=1}^L h(k, i, L) \{E_{\gamma}[\bar{\gamma}_1(i)]\bar{H}K_{\bar{x}}(i, s)\bar{H}^T E_{\gamma}[\bar{\gamma}_1^T(s)] + E_{\gamma}[\bar{\gamma}_2(i)]K_{\bar{v}}(i, s)E_{\gamma}[\bar{\gamma}_2^T(s)]\}. \end{aligned} \quad (23)$$

From

$$\begin{aligned} \bar{p}_1(k) &= E_{\gamma}[\bar{\gamma}_1(k)] = \begin{bmatrix} p_{01}(k)I_{m \times m} & p_{11}(k)I_{m \times m} & p_{21}(k)I_{m \times m} & \cdots & p_{\bar{N}1}(k)I_{m \times m} \end{bmatrix}, \\ \bar{p}_2(k) &= E_{\gamma}[\bar{\gamma}_2(k)] = E_{\gamma}[\bar{\gamma}_0(k) + \bar{\gamma}_1(k)] = \bar{p}_0(k) + \bar{p}_1(k) \\ &= \begin{bmatrix} (p_{00}(k) + p_{01}(k))I_{m \times m} & (p_{10}(k) + p_{11}(k))I_{m \times m} & (p_{20}(k) + p_{21}(k))I_{m \times m} & \cdots & (p_{\bar{N}0}(k) + p_{\bar{N}1}(k))I_{m \times m} \end{bmatrix}, \\ \bar{\gamma}_1(k) &= \begin{bmatrix} \gamma_{01}(k)I_{m \times m} & \gamma_{11}(k)I_{m \times m} & \gamma_{21}(k)I_{m \times m} & \cdots & \gamma_{\bar{N}1}(k)I_{m \times m} \end{bmatrix}, \\ \bar{\gamma}_0(k) &= \begin{bmatrix} \gamma_{00}(k)I_{m \times m} & \gamma_{10}(k)I_{m \times m} & \gamma_{20}(k)I_{m \times m} & \cdots & \gamma_{\bar{N}0}(k)I_{m \times m} \end{bmatrix}, \end{aligned} \quad (24)$$

(23) is written as

$$\begin{aligned} h(k, s, L) \{E_{\gamma}[\bar{\gamma}_1(s)\bar{H}K_{\bar{x}}(s, s)\bar{H}^T \bar{\gamma}_1^T(s)] + E_{\gamma}[\bar{\gamma}_2(s)K_{\bar{v}}(s, s)\bar{\gamma}_2^T(s)] \\ - E_{\gamma}[\bar{\gamma}_1(s)]\bar{H}K_{\bar{x}}(s, s)\bar{H}^T E_{\gamma}[\bar{\gamma}_1^T(s)] - E_{\gamma}[\bar{\gamma}_2(s)]K_{\bar{v}}(s, s)E_{\gamma}[\bar{\gamma}_2^T(s)]\} &= K_{\bar{x}}(k, s)H^T \bar{p}_1^T(s) \\ - \sum_{i=1}^L h(k, i, L) \{E_{\gamma}[\bar{\gamma}_1(i)]\bar{H}K_{\bar{x}}(i, s)\bar{H}^T E_{\gamma}[\bar{\gamma}_1^T(s)] + E_{\gamma}[\bar{\gamma}_2(i)]K_{\bar{v}}(i, s)E_{\gamma}[\bar{\gamma}_2^T(s)]\}. \end{aligned} \quad (25)$$

Consequently, the optimal impulse response function $h(k, s, L)$ satisfies

$$\begin{aligned} h(k, s, L)\bar{R}(s) &= K_{\bar{x}}(k, s)H^T \bar{p}_1^T(s) \\ - \sum_{i=1}^L h(k, i, L) \{ \bar{p}_1(i)\bar{H}K_{\bar{x}}(i, s)\bar{H}^T \bar{p}_1^T(s) + \bar{p}_2(i)K_{\bar{v}}(i, s)\bar{p}_2^T(s) \}, \end{aligned} \quad (26)$$

$$\begin{aligned} \bar{R}(s) &= E_{\gamma}[\bar{\gamma}_1(s)\bar{H}K_{\bar{x}}(s, s)\bar{H}^T \bar{\gamma}_1^T(s)] + E_{\gamma}[\bar{\gamma}_2(s)K_{\bar{v}}(s, s)\bar{\gamma}_2^T(s)] \\ &\quad - E_{\gamma}[\bar{\gamma}_1(s)]\bar{H}K_{\bar{x}}(s, s)\bar{H}^T E_{\gamma}[\bar{\gamma}_1^T(s)] - E_{\gamma}[\bar{\gamma}_2(s)]K_{\bar{v}}(s, s)E_{\gamma}[\bar{\gamma}_2^T(s)]. \end{aligned} \quad (27)$$

RLS Wiener estimation algorithms

Under the linear least-squares estimation problem of the signal $z(k)$ in section 2, [Theorem 1] shows the RLS Wiener fixed-point smoothing and filtering algorithms, which use the covariance information of the signal and observation noise.

Theorem 1

Let the auto-covariance function $K_x(k, s)$ of the state vector $x(k)$ be expressed by (9), and let the variance of white observation noise be R . Then, the RLS Wiener algorithms for the fixed-point smoothing at the fixed



point k and filtering estimate of the signal $z(k)$ consist of (28)-(41) in linear discrete-time stochastic systems with randomly delayed observations..

Fixed-point smoothing estimate of the signal $z(k)$: $\hat{z}(k, L)$

$$\hat{z}(k, L) = H\hat{x}(k, L) \quad (28)$$

Filtering estimate of the signal $z(L)$

$$\hat{z}(L, L) = H\hat{x}(L, L) \quad (29)$$

$$\hat{\bar{x}}(k, L) = \hat{\bar{x}}(k, L-1) + h(k, L, L)(y(L) - \bar{p}_1(L)\bar{H}\bar{\Phi}\hat{\bar{x}}(L-1, L-1) - \bar{p}_2(L)\bar{\Phi}_{\bar{v}}\hat{\bar{v}}(L-1, L-1)),$$

$$\hat{x}(k, L) = [0_{n \times n} \quad 0_{n \times n} \quad \cdots \quad 0_{n \times n} \quad I_{n \times n}] \hat{\bar{x}}(k, L),$$

$$\hat{\bar{x}}(k, L) = \begin{bmatrix} \hat{x}(k, L) \\ \hat{x}(k-1, L) \\ \vdots \\ \hat{x}(k-\bar{N}, L) \end{bmatrix} \quad (30)$$

$$\begin{aligned} h(k, L, L) = & [K_{\bar{x}}(k, k)(\bar{\Phi}^T)^{L-k} \bar{H}^T \bar{p}_1^T(L) - q_1(k, L-1)\bar{\Phi}^T \bar{H}^T \bar{p}_1^T(L) - q_2(k, L-1)\Phi_{\bar{v}}^T \bar{p}_2^T(L)] \\ & \times \{ \bar{R}(L) + [\bar{p}_1(L)\bar{H}K_{\bar{x}}(L, L) - \bar{p}_1(L)\bar{H}\bar{\Phi}S_{11}(L-1)\bar{\Phi}^T - \bar{p}_2(L)\Phi_{\bar{v}}S_{21}(L-1)\bar{\Phi}^T] \bar{H}^T \bar{p}_1^T(L) \\ & + [\bar{p}_2(L)K_{\bar{v}}(L, L) - \bar{p}_1(L)\bar{H}\bar{\Phi}S_{12}(L-1)\bar{\Phi}_{\bar{v}}^T - \bar{p}_2(L)\Phi_{\bar{v}}S_{22}(L-1)\Phi_{\bar{v}}^T] \bar{H}^T \bar{p}_2^T(L) \}^{-1} \end{aligned} \quad (31)$$

$$\begin{aligned} q_1(k, L) = & q_1(k, L-1)\bar{\Phi}^T + h(k, L, L)[\bar{p}_1(L)\bar{H}K_{\bar{x}}(L, L) - \bar{p}_1(L)\bar{H}\bar{\Phi}S_{11}(L-1)\bar{\Phi}^T \\ & - \bar{p}_2(L)\Phi_{\bar{v}}S_{21}(L-1)\bar{\Phi}^T], \\ q_1(k, k) = & S_{11}(k) \end{aligned} \quad (32)$$

$$\begin{aligned} q_2(k, L) = & q_2(k, L-1)\Phi_{\bar{v}}^T + h(k, L, L)[\bar{p}_2(L)K_{\bar{v}}(L, L) - \bar{p}_1(L)\bar{H}\bar{\Phi}S_{12}(L-1)\Phi_{\bar{v}}^T \\ & - \bar{p}_2(L)\Phi_{\bar{v}}S_{22}(L-1)\Phi_{\bar{v}}^T], \\ q_2(k, k) = & S_{12}(k) \end{aligned} \quad (33)$$

Filtering estimate of $\bar{x}(L)$: $\hat{\bar{x}}(L, L)$

$$\begin{aligned} \hat{\bar{x}}(L, L) = & \bar{\Phi}\hat{\bar{x}}(L-1, L-1) + G_1(L, L)(y(L) - \bar{p}_1(L)\bar{H}\bar{\Phi}\hat{\bar{x}}(L-1, L-1) \\ & - \bar{p}_2(L)\Phi_{\bar{v}}\hat{\bar{v}}(L-1, L-1)), \\ \hat{\bar{x}}(0, 0) = & 0 \end{aligned} \quad (34)$$

Filtering estimate of $\bar{v}(L)$: $\hat{\bar{v}}(L, L)$

$$\begin{aligned} \hat{\bar{v}}(L, L) = & \Phi_{\bar{v}}\hat{\bar{v}}(L-1, L-1) + G_2(L, L)(y(L) - \bar{p}_1(L)\bar{H}\bar{\Phi}\hat{\bar{x}}(L-1, L-1) \\ & - \bar{p}_2(L)\Phi_{\bar{v}}\hat{\bar{v}}(L-1, L-1)), \\ \hat{\bar{v}}(0, 0) = & 0 \end{aligned} \quad (35)$$

Auto-variance function of $\hat{\bar{x}}(L, L)$: $S_{11}(L) = E[\hat{\bar{x}}(L, L)\hat{\bar{x}}^T(L, L)]$



$$\begin{aligned}
S_{11}(L) &= \bar{\Phi} S_{11}(L-1) \bar{\Phi}^T + G_1(L, L) [\bar{p}_1(L) \bar{H} K_{\bar{x}}(L, L) - \bar{p}_1(L) \bar{H} \bar{\Phi} S_{11}(L-1) \bar{\Phi}^T \\
&\quad - \bar{p}_2(L) \Phi_{\bar{v}} S_{21}(L-1) \bar{\Phi}^T], \\
S_{11}(0) &= 0
\end{aligned} \tag{36}$$

Cross-variance function of $\hat{\bar{x}}(L, L)$ with $\hat{\bar{v}}(L, L)$: $S_{12}(L) = E[\hat{\bar{x}}(L, L) \hat{\bar{v}}^T(L, L)]$

$$\begin{aligned}
S_{12}(L) &= \bar{\Phi} S_{12}(L-1) \Phi_{\bar{v}}^T + G_1(L, L) [\bar{p}_2(L) K_{\bar{v}}(L, L) - \bar{p}_1(L) \bar{H} \bar{\Phi} S_{12}(L-1) \Phi_{\bar{v}}^T \\
&\quad - \bar{p}_2(L) \Phi_{\bar{v}} S_{22}(L-1) \Phi_{\bar{v}}^T], \\
S_{12}(0) &= 0
\end{aligned} \tag{37}$$

$$\begin{aligned}
S_{21}(L) &= \Phi_{\bar{v}} S_{21}(L-1) \bar{\Phi}^T + G_2(L, L) [\bar{p}_1(L) \bar{H} K_{\bar{x}}(L, L) - \bar{p}_1(L) \bar{H} \bar{\Phi} S_{11}(L-1) \bar{\Phi}^T \\
&\quad - \bar{p}_2(L) \Phi_{\bar{v}} S_{21}(L-1) \bar{\Phi}^T], \\
S_{21}(0) &= 0, S_{21}(L) = S_{12}^T(L)
\end{aligned} \tag{38}$$

Auto-variance function of $\hat{\bar{v}}(L, L)$: $S_{22}(L) = E[\hat{\bar{v}}(L, L) \hat{\bar{v}}^T(L, L)]$

$$\begin{aligned}
S_{22}(L) &= \Phi_{\bar{v}} S_{22}(L-1) \Phi_{\bar{v}}^T + G_2(L, L) [\bar{p}_2(L) K_{\bar{v}}(L, L) - \bar{p}_1(L) \bar{H} \bar{\Phi} S_{12}(L-1) \Phi_{\bar{v}}^T \\
&\quad - \bar{p}_2(L) \Phi_{\bar{v}} S_{22}(L-1) \Phi_{\bar{v}}^T], \\
S_{22}(0) &= 0
\end{aligned} \tag{39}$$

$$\begin{aligned}
G_1(L, L) &= [K_{\bar{x}}(L, L) \bar{H}^T \bar{p}_1^T(L) - \bar{\Phi} S_{11}(L-1) \bar{\Phi}^T \bar{H}^T \bar{p}_1^T(L) - \bar{\Phi} S_{12}(L-1) \Phi_{\bar{v}}^T \bar{p}_2^T(L)] \\
&\quad \times \{ \bar{R}(L) + [\bar{p}_1(L) \bar{H} K_{\bar{x}}(L, L) - \bar{p}_1(L) \bar{H} \bar{\Phi} S_{11}(L-1) \bar{\Phi}^T - \bar{p}_2(L) \Phi_{\bar{v}} S_{21}(L-1) \bar{\Phi}^T] \bar{H}^T \bar{p}_1^T(L) \\
&\quad + [\bar{p}_2(L) K_{\bar{v}}(L, L) - \bar{p}_1(L) \bar{H} \bar{\Phi} S_{12}(L-1) \Phi_{\bar{v}}^T - \bar{p}_2(L) \Phi_{\bar{v}} S_{22}(L-1) \Phi_{\bar{v}}^T] \bar{H}^T \bar{p}_2^T(L) \}^{-1}
\end{aligned} \tag{40}$$

$$\begin{aligned}
G_2(L, L) &= [K_{\bar{v}}(L, L) \bar{p}_2^T(L) - \Phi_{\bar{v}} S_{21}(L-1) \bar{\Phi}^T \bar{H}^T \bar{p}_1^T(L) - \Phi_{\bar{v}} S_{22}(L-1) \Phi_{\bar{v}}^T \bar{p}_2^T(L)] \\
&\quad \times \{ \bar{R}(L) + [\bar{p}_1(L) \bar{H} K_{\bar{x}}(L, L) - \bar{p}_1(L) \bar{H} \bar{\Phi} S_{11}(L-1) \bar{\Phi}^T - \bar{p}_2(L) \Phi_{\bar{v}} S_{21}(L-1) \bar{\Phi}^T] \bar{H}^T \bar{p}_1^T(L) \\
&\quad + [\bar{p}_2(L) K_{\bar{v}}(L, L) - \bar{p}_1(L) \bar{H} \bar{\Phi} S_{12}(L-1) \Phi_{\bar{v}}^T - \bar{p}_2(L) \Phi_{\bar{v}} S_{22}(L-1) \Phi_{\bar{v}}^T] \bar{H}^T \bar{p}_2^T(L) \}^{-1}
\end{aligned} \tag{41}$$

$$\begin{aligned}
\bar{R}(L) &= E_{\gamma} [\bar{\gamma}_1(L) \bar{H} K_{\bar{x}}(L, L) \bar{H}^T \bar{\gamma}_1^T(L)] + E_{\gamma} [\bar{\gamma}_2(L) K_{\bar{v}}(L, L) \bar{\gamma}_2^T(L)] \\
&\quad - E_{\gamma} [\bar{\gamma}_1(L) \bar{H} K_{\bar{x}}(L, L) \bar{H}^T E_{\gamma} [\bar{\gamma}_1^T(L)] - E_{\gamma} [\bar{\gamma}_2(L) K_{\bar{v}}(L, L) E_{\gamma} [\bar{\gamma}_2^T(L)]]
\end{aligned} \tag{42}$$

Derivation of [Theorem 1] is same as that proposed in [5] for the filtering and fixed-point smoothing problem from randomly delayed observations by one sampling time. So, the proof of [Theorem 1] is omitted.

From [Theorem 1], it is found that the filtering error variance function of $\bar{z}(L)$ is given by

$$P_{\bar{z}}(L) = \bar{H} \bar{H}^T (K_{\bar{x}}(L, L) - S_{11}(L)) \bar{H}^T \bar{H}^T, \quad \bar{H} = \begin{bmatrix} I_{m \times m} & 0_{m \times m} & \cdots & 0_{m \times m} \end{bmatrix}. \tag{43}$$

A numerical simulation example

Let an m-dimensional observation equation be given by

$$\begin{aligned}
y(k) &= \gamma_{01}(k) \bar{y}(k) + \gamma_{11}(k) \bar{y}(k-1) + \gamma_{21}(k) \bar{y}(k-2) \\
&\quad + \gamma_{00}(k) v(k) + \gamma_{10}(k) v(k-1) + \gamma_{20}(k) v(k-2),
\end{aligned}$$



$$\begin{aligned}
\Pr\{\gamma_{01}(k)\} &= \Pr\{\gamma_{01}(k)^2\} = p_{01}, \Pr\{\gamma_{11}(k)\} = \Pr\{\gamma_{11}(k)^2\} = p_{11}, \\
\Pr\{\gamma_{21}(k)\} &= \Pr\{\gamma_{21}(k)^2\} = p_{21}, \Pr\{\gamma_{00}(k)\} = \Pr\{\gamma_{00}(k)^2\} = p_{00}, \\
\Pr\{\gamma_{10}(k)\} &= \Pr\{\gamma_{10}(k)^2\} = p_{10}, \Pr\{\gamma_{20}(k)\} = \Pr\{\gamma_{20}(k)^2\} = p_{20} \\
\bar{y}(k) &= z(k) + v(k), z(k) = Hx(k).
\end{aligned} \tag{44}$$

Here, p_{01} , p_{11} , p_{21} , p_{00} , p_{10} and p_{20} are the probabilities shown in Table 1.

Table 1 Probabilities for the Bernoulli variables $\gamma_{01}(k)$, $\gamma_{11}(k)$, $\gamma_{21}(k)$, $\gamma_{00}(k)$, $\gamma_{10}(k)$ and $\gamma_{20}(k)$.

Cases of delay	Probability of the observation including both signal and observation noise	Probability of the observation not including signal and consisting of only observation noise
No delay	$\Pr\{\gamma_{01}(k) = 1\} = p_{01}(k) = 0.8820$	$\Pr\{\gamma_{00}(k) = 1\} = p_{00}(k) = 0.0180$
One-step delay	$\Pr\{\gamma_{11}(k) = 1\} = p_{11}(k) = 0.0570$	$\Pr\{\gamma_{10}(k) = 1\} = p_{10}(k) = 0.0030$
Two-steps delay	$\Pr\{\gamma_{21}(k) = 1\} = p_{21}(k) = 0.0360$	$\Pr\{\gamma_{20}(k) = 1\} = p_{20}(k) = 0.0040$

Let the observation noise $v(k)$ be zero-mean white Gaussian process with the variance R , $N(0, R)$. R , $N(0, R)$. (42) is also written as

$$\begin{aligned}
y(k) &= \bar{\gamma}_1(k) \tilde{y}(k) + \bar{\gamma}_0(k) \bar{v}(k) = \bar{\gamma}_1(k) \bar{H} \bar{x}(k) + \bar{\gamma}_2(k) \bar{v}(k), \\
\bar{\gamma}_1(k) &= [\gamma_{01}(k) \quad \gamma_{11}(k) \quad \gamma_{21}(k)], \bar{\gamma}_0(k) = [\gamma_{00}(k) \quad \gamma_{10}(k) \quad \gamma_{20}(k)], \\
\bar{\gamma}_2(k) &= [\gamma_{01}(k) + \gamma_{00}(k) \quad \gamma_{10}(k) + \gamma_{11}(k) \quad \gamma_{20}(k) + \gamma_{21}(k)], \\
\bar{p}_1(k) &= [p_{01}(k) \quad p_{11}(k) \quad p_{21}(k)], \bar{p}_0(k) = [p_{00}(k) \quad p_{10}(k) \quad p_{20}(k)], \\
\bar{p}_2(k) &= [p_{01}(k) + p_{00}(k) \quad p_{10}(k) + p_{11}(k) \quad p_{20}(k) + p_{21}(k)], \\
\tilde{y}(k) &= [\bar{y}(k) \quad \bar{y}(k-1) \quad \bar{y}(k-2)]^T, \\
\bar{v}(k) &= [v(k) \quad v(k-1) \quad v(k-2)]^T \\
y(k) &= \bar{\gamma}(k) \bar{z}(k) + \bar{v}(k), \\
\bar{z}(k) &= \begin{bmatrix} z(k) \\ z(k-1) \\ z(k-2) \end{bmatrix} = \begin{bmatrix} Hx(k) \\ Hx(k-1) \\ Hx(k-2) \end{bmatrix} = \bar{H} \bar{x}(k).
\end{aligned} \tag{45}$$

Let the signal $z(k)$ be generated by the second-order AR model.

$$\begin{aligned}
z(k+1) &= -a_1 z(k) - a_2 z(k-1) + w(k), E[w(k)w(s)] = \sigma^2 \delta_K(k-s), \\
a_1 &= -0.1, a_2 = -0.8, \sigma = 0.5.
\end{aligned} \tag{46}$$

The state-space model for $z(k)$ is given by

$$\begin{aligned}
z(k) &= Hx(k) = x_1(k), H = [1 \quad 0], x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \\
\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(k).
\end{aligned} \tag{47}$$



Hence,

\bar{H} and $\bar{x}(k)$ are given by

$$\bar{H} = \begin{bmatrix} H & 0 & 0 \\ 0 & H & 0 \\ 0 & 0 & H \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \bar{x}(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_1(k-1) \\ x_2(k-1) \\ x_1(k-2) \\ x_2(k-2) \end{bmatrix} = \begin{bmatrix} x_1(k) \\ x_1(k+1) \\ x_1(k-1) \\ x_1(k) \\ x_1(k-2) \\ x_1(k-1) \end{bmatrix}. \quad (48)$$

From $\bar{x}(k)$, the signal $z(k)$ is calculated by

$$z(k) = [1 \ 0 \ 0 \ 0 \ 0 \ 0] \bar{x}(k) = x_1(k). \quad (49)$$

The auto-covariance function of the signal $z(k)$ is given by [9]

$$\begin{aligned} K(0) &= \sigma^2, \\ K(m) &= \sigma^2 \{ \alpha_1(\alpha_2^2 - 1)\alpha_1^m / [(\alpha_2 - \alpha_1)(\alpha_2\alpha_1 + 1)] \\ &\quad - \alpha_2(\alpha_1^2 - 1)\alpha_2^m / [(\alpha_2 - \alpha_1)(\alpha_1\alpha_2 + 1)] \}, 0 < m, \\ \alpha_1, \alpha_2 &= \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}. \end{aligned} \quad (50)$$

From

$$K_x(k, k) = \begin{bmatrix} K(0) & K(1) \\ K(1) & K(0) \end{bmatrix}, \Phi = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}, \quad (51)$$

$\bar{\Phi}$ and $K_{\bar{x}}(0)$ are given by

$$\bar{\Phi} = \begin{bmatrix} \Phi & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -a_2 & -a_1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad (52)$$

$$\begin{aligned} K_{\bar{x}}(0) &= \begin{bmatrix} K_x(0) & \Phi K_x(0) & \Phi^2 K_x(0) \\ K_x(0)\Phi^T & K_x(0) & \Phi K_x(0) \\ K_x(0)(\Phi^T)^2 & K_x(0)\Phi^T & K_x(0) \end{bmatrix}, \\ &= \begin{bmatrix} K(0) & K(1) & K(1) & K(0) & -a_2 K(0) - a_1 K(1) & -a_2 K(1) - a_1 K(0) \\ K(1) & K(0) & -a_2 K(0) - a_1 K(1) & -a_2 K(1) - a_1 K(0) & -a_2 K(1) + a_1(a_2 K(0) + a_1 K(1)) & -a_2 K(0) + a_1(a_2 K(1) + a_1 K(0)) \\ K(1) & -a_2 K(0) - a_1 K(1) & K(0) & K(1) & K(1) & K(0) \\ K(0) & -a_2 K(1) - a_1 K(0) & K(1) & K(0) & -a_2 K(0) - a_1 K(1) & -a_2 K(1) - a_1 K(0) \\ -a_2 K(0) - a_1 K(1) & -a_2 K(1) + a_1(a_2 K(0) + a_1 K(1)) & K(1) & -a_2 K(0) - a_1 K(1) & K(0) & K(1) \\ -a_2 K(1) - a_1 K(0) & -a_2 K(0) + a_1(a_2 K(1) + a_1 K(0)) & K(0) & -a_2 K(1) - a_1 K(0) & K(1) & K(0) \end{bmatrix}, \end{aligned}$$

$$K(0) = 0.25, K(1) = 0.125. \quad (53)$$

From (44), (45), \bar{H} in (48), and (53), $\bar{R}(L)$ is calculated as



$$\begin{aligned}
\bar{R}(L) &= E_{\gamma}[\bar{\gamma}_1(L)\bar{H}K_{\bar{x}}(L,L)\bar{H}^T\bar{\gamma}_1^T(L)] + E_{\gamma}[\bar{\gamma}_2(L)K_{\bar{v}}(L,L)\bar{\gamma}_2^T(L)] \\
&\quad - E_{\gamma}[\bar{\gamma}_1(L)]\bar{H}K_{\bar{x}}(L,L)\bar{H}^TE_{\gamma}[\bar{\gamma}_1^T(L)] - E_{\gamma}[\bar{\gamma}_2(L)]K_{\bar{v}}(L,L)E_{\gamma}[\bar{\gamma}_2^T(L)] \\
&= p_{01}(K(0) + p_{11}K(1) + p_{21}(a_2K(0) - a_1K(1))) \\
&\quad + p_{11}(p_{01}K(1) + K(0) + p_{21}K(1)) + (p_{00} + p_{10} + p_{20})R \\
&\quad - p_{01}(p_{01}K(0) + p_{11}K(1) + p_{21}(a_2K(0) - a_1K(1))) \\
&\quad - p_{11}(p_{01}K(1) + p_{11}K(0) + p_{21}K(1)) - (p_{00}^2 + p_{10}^2 + p_{20}^2)R.
\end{aligned} \tag{54}$$

Substituting \bar{H} , $\bar{\Phi}$, $K_{\bar{x}}(L,L) = K_{\bar{x}}(0)$, $\bar{p}_1(L)$, $\bar{p}_2(L)$ and $\bar{R}(L)$ into the RLS Wiener estimation algorithms in [Theorem 1], the filtering and fixed-point smoothing estimates are calculated recursively.

Fig.1 illustrates the fixed-point smoothing estimate $\hat{z}(k, k+5)$ vs. k for the white Gaussian observation noise $N(0, 0.3^2)$. Fig.2 illustrates the MSVs (mean-square values) of the filtering errors $z(k) - \hat{z}(k, k)$ and the fixed-point smoothing errors $z(k) - \hat{z}(k, k + Lag)$ vs. Lag , $0 \leq Lag \leq 5$, for the white Gaussian observation noises $N(0, 0.1^2)$, $N(0, 0.2^2)$, $N(0, 0.3^2)$ and $N(0, 0.4^2)$. For $Lag = 0$, the MSV of the filtering errors $z(k) - \hat{z}(k, k)$ is shown. Fig.3 indicates the tendency, as Lag increases, that the MSVs decrease gradually and the estimation accuracy is improved by the fixed-point smoother. For the white Gaussian observation noise with larger variance, the MSVs of the filtering errors and the fixed-point smoothing errors increase and the estimation accuracy becomes degraded. Here, The MSVs of the fixed-point smoothing and filtering errors are evaluated by $\sum_{i=1}^{2000} (z(k) - \hat{z}(k, k + Lag))^2 / 2000$ and $\sum_{i=1}^{2000} (z(k) - \hat{z}(k, k))^2 / 2000$.

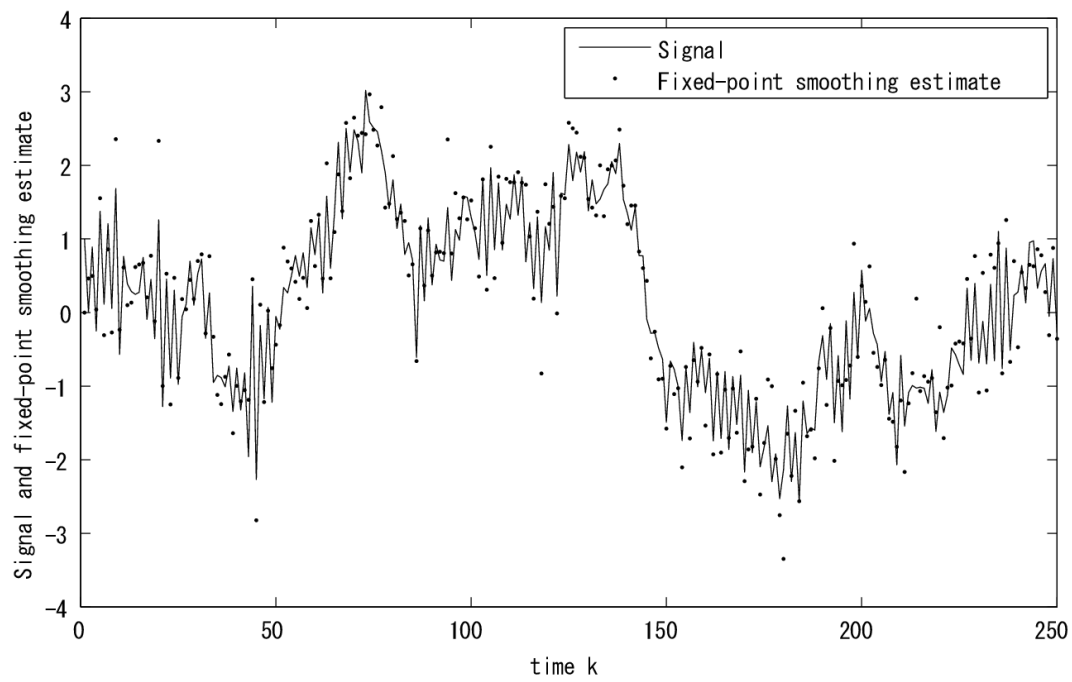


Fig.1 Fixed-point smoothing estimate $\hat{z}(k, k+5)$ vs. k for the white Gaussian observation noise $N(0, 0.3^2)$.

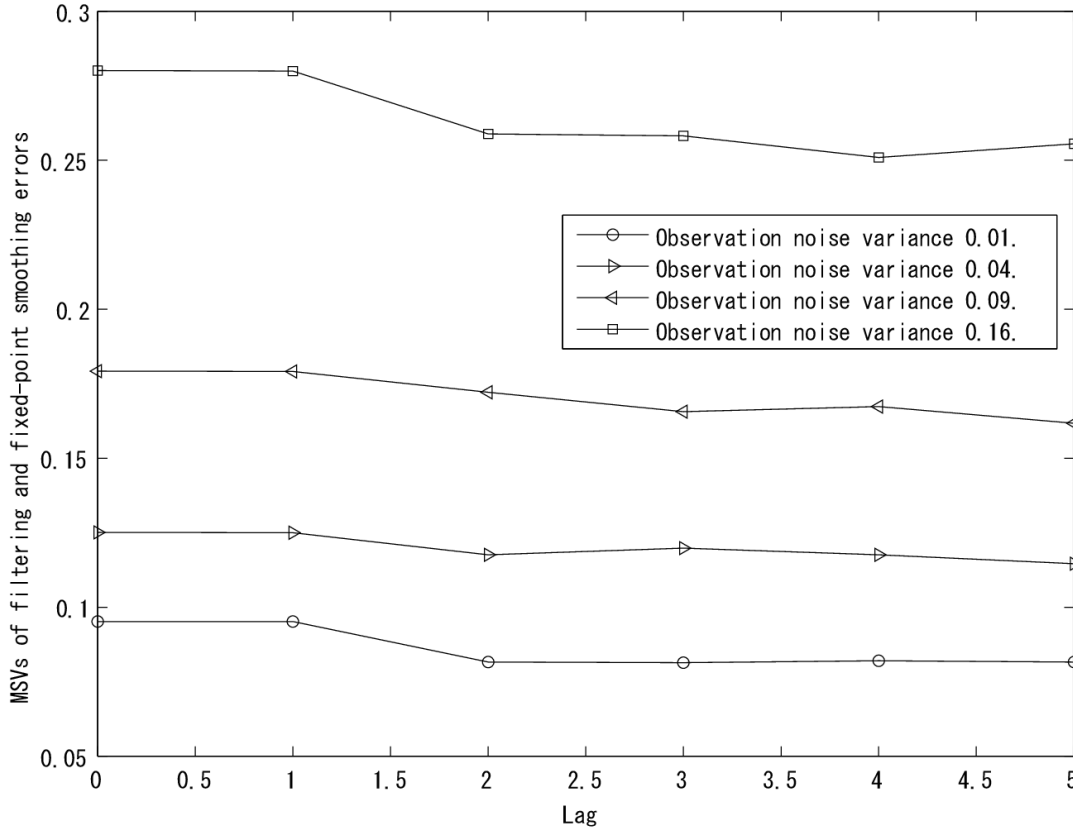


Fig.2 MSVs of the filtering errors $z(k) - \hat{z}(k, k)$ and the fixed-point smoothing errors $z(k) - \hat{z}(k, k + Lag)$ vs. Lag , $0 \leq Lag \leq 5$, for the white Gaussian observation noises $N(0, 0.1^2)$, $N(0, 0.2^2)$, $N(0, 0.3^2)$ and $N(0, 0.4^2)$.



Conclusions

This paper proposed the RLS Wiener fixed-point smoother and filter from randomly delayed observed values by multiple sampling times or uncertain observations in linear discrete-time stochastic systems. Some numerical simulation results have shown that the devised estimators have feasible estimation characteristics.

Since the RLS Wiener estimators do not use the information of the variance Q of the input noise and the input matrix B in the state equation (11), in comparison with the estimation technique [1]-[3] with additional information of Q and B to those used in the RLS Wiener estimators, the proposed RLS Wiener estimators do not incur estimation degradation influenced by the model uncertainty on Q and B .

In this paper, the RLS Wiener fixed-point smoother and filter are designed from observations with random and multiple delays in linear discrete-time stochastic systems. The probability of the arrival of the observed value $y(k)$ on time k is p_1 and the probabilities of the arrival of the observed value $y(k-(j-1))$, $j=2,3,\dots,N$, on time k is p_j . A numerical simulation example has shown that the proposed estimation technique with the randomly delayed observed values is feasible.

Appendix (Proof of Theorem 1)

From (25), the impulse response function $h(k, s, L)$ satisfies

$$h(k, s, L)\bar{R}(s) = K_{\bar{x}}(k, s)H^T \bar{p}_1^T(s) - \sum_{i=1}^L h(k, i, L)[\bar{p}_1(i)\bar{H}K_{\bar{x}}(i, s)\bar{H}^T \bar{p}_1^T(s) + \bar{p}_2(i)K_{\bar{v}}(i, s)\bar{p}_2^T(s)]. \quad (\text{A-1})$$

Subtracting $h(k, s, L-1)\bar{R}(s)$ from $h(k, s, L)\bar{R}(s)$, we have

$$(h(k, s, L) - h(k, s, L-1))\bar{R}(s) = -h(k, L, L)[\bar{p}_1(L)\bar{H}K_{\bar{x}}(L, s)\bar{H}^T \bar{p}_1^T(s) + \bar{p}_2(L)K_{\bar{v}}(L, s)\bar{p}_2^T(s)] - \sum_{i=1}^{L-1} (h(k, i, L) - h(k, i, L-1))[\bar{p}_1(i)\bar{H}K_{\bar{x}}(i, s)\bar{H}^T \bar{p}_1^T(s) + \bar{p}_2(i)K_{\bar{v}}(i, s)\bar{p}_2^T(s)].$$

Introducing

$$J_1(s, L-1)\bar{R}(s) = \bar{\Phi}^{-s} K_{\bar{x}}(s, s)\bar{H}^T \bar{p}_1^T(s) - \sum_{i=1}^{L-1} J_1(i, L-1)[\bar{p}_1(i)\bar{H}K_{\bar{x}}(i, s)\bar{H}^T \bar{p}_1^T(s) + \bar{p}_2(i)K_{\bar{v}}(i, s)\bar{p}_2^T(s)], \quad (\text{A-2})$$

and

$$J_2(s, L-1)\bar{R}(s) = \Phi_{\bar{v}}^{-s} K_{\bar{v}}(s, s)\bar{H}^T \bar{p}_2^T(s) - \sum_{i=1}^{L-1} J_2(i, L-1)[\bar{p}_1(i)\bar{H}K_{\bar{x}}(i, s)\bar{H}^T \bar{p}_1^T(s) + \bar{p}_2(i)K_{\bar{v}}(i, s)\bar{p}_2^T(s)], \quad (\text{A-3})$$

we obtain

$$h(k, s, L) - h(k, s, L-1) = -h(k, L, L)[\bar{p}_1(L)\bar{H}\bar{\Phi}^L J_1(s, L-1) + \bar{p}_2(L)\Phi_{\bar{v}}^L J_2(s, L-1)]. \quad (\text{A-4})$$

Subtracting $J_1(s, L-1)\bar{R}(s)$ from $J_1(s, L)\bar{R}(s)$, we have



$$\begin{aligned}
& (J_1(s, L) - J_1(s, L-1))\bar{R}(s) = -J_1(L, L)[\bar{p}_1(L)\bar{H}\bar{K}_{\bar{x}}(L, s)\bar{H}^T\bar{p}_1^T(s) \\
& + \bar{p}_2(L)K_{\bar{v}}(L, s)\bar{p}_2^T(s)] \\
& - \sum_{i=1}^{L-1} (J_1(i, L) - J_1(i, L-1))[\bar{p}_1(i)\bar{H}\bar{K}_{\bar{x}}(i, s)\bar{H}^T\bar{p}_1^T(s) + \bar{p}_2(i)K_{\bar{v}}(i, s)\bar{p}_2^T(s)].
\end{aligned} \tag{A-5}$$

From (A-2) and (A-3), we obtain

$$J_1(s, L) - J_1(s, L-1) = -J_1(L, L)[\bar{p}_1(L)\bar{H}\bar{\Phi}^L J_1(s, L-1) + \bar{p}_2(L)\Phi_{\bar{v}}^L J_2(s, L-1)]. \tag{A-6}$$

Similarly, from (A-3), we obtain

$$J_2(s, L) - J_2(s, L-1) = -J_2(L, L)[\bar{p}_1(L)\bar{H}\bar{\Phi}^L J_1(s, L-1) + \bar{p}_2(L)\Phi_{\bar{v}}^L J_2(s, L-1)]. \tag{A-7}$$

From (A-2), $J_1(L-1, L-1)$ satisfies

$$\begin{aligned}
& J_1(L-1, L-1)\bar{R}(L-1) = \bar{\Phi}^{-(L-1)}K_{\bar{x}}(L-1, L-1)\bar{H}^T\bar{p}_1^T(L-1) \\
& - \sum_{i=1}^L J_1(i, L-1)[\bar{p}_1(i)\bar{H}\bar{K}_{\bar{x}}(i, L-1)\bar{H}^T\bar{p}_1^T(L-1) + \bar{p}_2(i)K_{\bar{v}}(i, L-1)\bar{p}_2^T(L-1)].
\end{aligned}$$

Using (A-5) and (A-10), and introducing

$$r_{11}(L-1) = \sum_{i=1}^{L-1} J_1(i, L-1)\bar{p}_1(i)\bar{H}\bar{B}(i), \tag{A-8}$$

$$r_{12}(L-1) = \sum_{i=1}^{L-1} J_1(i, L-1)\bar{p}_2(i)B_V(i), \tag{A-9}$$

we obtain

$$\begin{aligned}
& J_1(L-1, L-1)\bar{R}(L-1) = \bar{\Phi}^{-(L-1)}K_{\bar{x}}(L-1, L-1)\bar{H}^T\bar{p}_1^T(L-1) \\
& - r_{11}(L-1)\bar{A}^T(L-1)\bar{H}^T\bar{p}_1^T(L-1) - r_{12}(L-1)A_V^T(L-1)\bar{p}_2^T(L-1).
\end{aligned} \tag{A-10}$$

Similarly, using (A-3), (A-5) and (A-10), and introducing

$$r_{21}(L-1) = \sum_{i=1}^{L-1} J_2(i, L-1)\bar{p}_1(i)\bar{H}\bar{B}(i), \tag{A-11}$$

$$r_{22}(L-1) = \sum_{i=1}^{L-1} J_2(i, L-1)\bar{p}_2(i)B_V(i), \tag{A-12}$$

we obtain

$$\begin{aligned}
& J_2(L-1, L-1)\bar{R}(L-1) = \Phi_{\bar{v}}^{-(L-1)}K_{\bar{v}}(L-1, L-1)\bar{H}^T\bar{p}_2^T(L-1) \\
& - r_{21}(L-1)\bar{A}^T(L-1)\bar{H}^T\bar{p}_1^T(L-1) - r_{22}(L-1)A_V^T(L-1)\bar{p}_2^T(L-1).
\end{aligned} \tag{A-13}$$

Subtracting $r_{11}(L-1)$ from $r_{11}(L)$ and using (A-6), (A-8) and (A-11), we obtain

$$\begin{aligned}
& r_{11}(L-1) - r_{11}(L-2) = J_1(L-1, L-1)[\bar{p}_1(L-1)\bar{B}(L-1) \\
& - \bar{p}_1(L-1)\bar{H}\bar{\Phi}^{L-1}r_{11}(L-2) - \bar{p}_2(L-1)\Phi_{\bar{v}}^{L-1}r_{21}(L-2)].
\end{aligned} \tag{A-14}$$

Subtracting $r_{12}(L-1)$ from $r_{12}(L)$ and using (A-6), (A-9) and (A-12), we obtain

$$\begin{aligned}
& r_{12}(L-1) - r_{12}(L-2) = J_1(L-1, L-1)[\bar{p}_2(L-1)\bar{B}_V(L-1) \\
& - \bar{p}_1(L-1)\bar{H}\bar{\Phi}^{L-1}r_{12}(L-2) - \bar{p}_2(L-1)\Phi_{\bar{v}}^{L-1}r_{22}(L-2)].
\end{aligned} \tag{A-15}$$



Subtracting $r_{21}(L-1)$ from $r_{21}(L)$ and using (A-7), (A-8) and (A-11), we obtain

$$\begin{aligned} r_{21}(L-1) - r_{21}(L-2) &= J_2(L-1, L-1)[\bar{p}_1(L-1)\bar{B}(L-1) \\ &- \bar{p}_1(L-1)\bar{H}\Phi^{L-1}r_{11}(L-2) - \bar{p}_2(L-1)\Phi_{\bar{V}}^{L-1}r_{21}(L-2)]. \end{aligned} \quad (\text{A-16})$$

Subtracting $r_{22}(L-1)$ from $r_{22}(L)$ and using (A-7), (A-8) and (A-11), we obtain

$$\begin{aligned} r_{22}(L-1) - r_{22}(L-2) &= J_2(L-1, L-1)[\bar{p}_1(L-1)\bar{B}(L-1) \\ &- \bar{p}_1(L-1)\bar{H}\Phi^{L-1}r_{11}(L-2) - \bar{p}_2(L-1)\Phi_{\bar{V}}^{L-1}r_{21}(L-2)]. \end{aligned} \quad (\text{A-17})$$

Let us introduce the functions

$$\begin{aligned} S_{11}(L) &= \bar{\Phi}^L r_{11}(L)(\bar{\Phi}^T)^L, S_{12}(L) = \bar{\Phi}^L r_{12}(L)(\Phi_{\bar{V}}^T)^L, \\ S_{21}(L) &= \Phi_{\bar{V}}^L r_{21}(L)(\bar{\Phi}^T)^L, S_{22}(L) = \Phi_{\bar{V}}^L r_{22}(L)(\Phi_{\bar{V}}^T)^L. \end{aligned} \quad (\text{A-18})$$

From (A-14) and (A-18), we obtain

$$\begin{aligned} S_{11}(L) &= \bar{\Phi}S_{11}(L-1)\bar{\Phi}^T + G_1(L, L)[\bar{p}_1(L)K_{\bar{x}}(L, L) - \bar{p}_1(L)\bar{H}\bar{\Phi}S_{11}(L-1)\bar{\Phi}^T \\ &- \bar{p}_2(L)\Phi_{\bar{V}}S_{21}(L-1)\bar{\Phi}^T], S_{11}(0) = 0. \end{aligned} \quad (\text{A-19})$$

From (A-15) and (A-18), we obtain

$$\begin{aligned} S_{12}(L) &= \bar{\Phi}S_{12}(L-1)\Phi_{\bar{V}}^T + G_1(L, L)[\bar{p}_2(L)K_{\bar{V}}(L, L) - \bar{p}_1(L)\bar{H}\bar{\Phi}S_{12}(L-1)\Phi_{\bar{V}}^T \\ &- \bar{p}_2(L)\Phi_{\bar{V}}S_{22}(L-1)\Phi_{\bar{V}}^T], S_{12}(0) = 0. \end{aligned} \quad (\text{A-20})$$

From (A-16) and (A-18), we obtain

$$\begin{aligned} S_{21}(L) &= \Phi_{\bar{V}}S_{21}(L-1)\bar{\Phi}^T + G_2(L, L)[\bar{p}_1(L)\bar{H}K_{\bar{x}}(L, L) - \bar{p}_1(L)\bar{H}\bar{\Phi}S_{11}(L-1)\bar{\Phi}^T \\ &- \bar{p}_2(L)\Phi_{\bar{V}}S_{21}(L-1)\bar{\Phi}^T], S_{21}(0) = 0. \end{aligned} \quad (\text{A-21})$$

From (A-17) and (A-18), we obtain

$$\begin{aligned} S_{22}(L) &= \Phi_{\bar{V}}S_{22}(L-1)\Phi_{\bar{V}}^T + G_2(L, L)[\bar{p}_2(L)K_{\bar{V}}(L, L) - \bar{p}_1(L)\bar{H}\bar{\Phi}S_{12}(L-1)\Phi_{\bar{V}}^T \\ &- \bar{p}_2(L)\Phi_{\bar{V}}S_{22}(L-1)\Phi_{\bar{V}}^T], S_{22}(0) = 0. \end{aligned} \quad (\text{A-22})$$

Let us introduce the functions

$$G_1(L, L) = \bar{\Phi}^L J_1(L, L), G_2(L, L) = \Phi_{\bar{V}}^L J_2(L, L). \quad (\text{A-23})$$

Substituting (A-10) into (A-23) and using (A-18), we obtain

$$G_1(L, L) = [K_{\bar{x}}(L, L)\bar{H}^T \bar{p}_1^T(L) - S_{11}(L)\bar{H}^T \bar{p}_1^T(L) - S_{12}(L)\bar{p}_2^T(L)]\bar{R}^{-1}(L). \quad (\text{A-24})$$

Substituting (A-13) into (A-23) and using (A-18), we obtain

$$G_2(L, L) = [K_{\bar{V}}(L, L)\bar{p}_2^T(L) - S_{21}(L)\bar{H}^T \bar{p}_1^T(L) - S_{22}(L)\bar{p}_2^T(L)]\bar{R}^{-1}(L). \quad (\text{A-25})$$

Substituting (A-19) and (A-20) into (A-24), we obtain (40). Substituting (A-21) and (A-22) into (A-25), we obtain (41).

From (26), $h(k, L, L)$ satisfies



$$\begin{aligned}
h(k, L, L)\bar{R}(L) &= K_{\bar{x}}(k, L)\bar{H}^T \bar{p}_1^T(L) \\
&- \sum_{i=1}^L h(k, i, L) \{ \bar{p}_1(i) \bar{H} K_{\bar{x}}(i, L) \bar{H}^T \bar{p}_1^T(L) + \bar{p}_2(i) K_{\bar{v}}(i, L) \bar{p}_2^T(L) \} \\
&= \bar{B}(k) \bar{A}^T(L) \bar{H}^T \bar{p}_1^T(L) - \sum_{i=1}^L h(k, i, L) \bar{p}_1(i) \bar{H} \bar{B}(i) \bar{A}^T(L) \bar{H}^T \bar{p}_1^T(L) \\
&- \sum_{i=1}^L h(k, i, L) \bar{p}_2(i) B_v(i) A_v^T(L) \bar{p}_2^T(L).
\end{aligned} \tag{A-26}$$

Introducing the functions

$$P_1(k, L) = \sum_{i=1}^L h(k, i, L) \bar{p}_1(i) \bar{H} \bar{B}(i), \tag{A-27}$$

$$P_2(k, L) = \sum_{i=1}^L h(k, i, L) \bar{p}_2(i) B_v(i), \tag{A-28}$$

we rewrite (A-26) as

$$\begin{aligned}
h(k, L, L)\bar{R}(L) &= K_{\bar{x}}(k, k)(\bar{\Phi}^T)^{L-k} \bar{H}^T \bar{p}_1^T(L) - P_1(k, L) \bar{A}^T(L) \bar{H}^T \bar{p}_1^T(L) \\
&- P_2(k, L) A_v^T(L) \bar{p}_2^T(L).
\end{aligned} \tag{A-29}$$

Subtracting $P_1(k, L-1)$ from $P_1(k, L)$ and using (A-4), we have

$$\begin{aligned}
P_1(k, L) &= P_1(k, L-1) + h(k, L, L) [\bar{p}_1(L) \bar{H} \bar{B}(L) - \bar{p}_1(L) \bar{H} \bar{\Phi}^L r_{11}(L-1) \\
&- \bar{p}_2(L) \Phi_{\bar{v}}^L r_{21}(L-1)].
\end{aligned} \tag{A-30}$$

Subtracting $P_2(k, L-1)$ from $P_2(k, L)$ and using (A-4), we have

$$\begin{aligned}
P_2(k, L) &= P_2(k, L-1) + h(k, L, L) [\bar{p}_2(L) K_v(L, L) - \bar{p}_1(L) \bar{H} \bar{\Phi}^L r_{12}(L-1) \\
&- \bar{p}_2(L) \Phi_{\bar{v}}^L r_{22}(L-1)].
\end{aligned} \tag{A-31}$$

Introducing

$$\begin{aligned}
q_1(k, L) &= P_1(k, L)(\bar{\Phi}^T)^L, \\
q_2(k, L) &= P_2(k, L)(\Phi_{\bar{v}}^T)^L,
\end{aligned} \tag{A-32}$$

From (A-30) and (A-31), we obtain

$$\begin{aligned}
q_1(k, L) &= q_1(k, L-1) \bar{\Phi}^T + h(k, L, L) [\bar{p}_1(L) \bar{H} K_{\bar{x}}(L, L) \\
&- \bar{p}_1(L) \bar{H} \bar{\Phi} S_{11}(L-1) \bar{\Phi}^T - \bar{p}_2(L) \Phi S_{21}(L-1) \bar{\Phi}^T],
\end{aligned} \tag{A-33}$$

$$\begin{aligned}
q_2(k, L) &= q_2(k, L-1) \bar{\Phi}_{\bar{v}}^T + h(k, L, L) [\bar{p}_2(L) K_{\bar{v}}(L, L) \\
&- \bar{p}_1(L) \bar{H} \bar{\Phi} S_{12}(L-1) \bar{\Phi}_{\bar{v}}^T - \bar{p}_2(L) \Phi_{\bar{v}} S_{22}(L-1) \bar{\Phi}_{\bar{v}}^T].
\end{aligned} \tag{A-34}$$

In (A-1), putting $L = s$, we have

$$\begin{aligned}
h(k, s, k) \bar{R}(s) &= K_{\bar{x}}(k, s) H^T \bar{p}_1^T(s) \\
&- \sum_{i=1}^k h(k, i, k) [\bar{p}_1(i) \bar{H} K_{\bar{x}}(i, s) \bar{H}^T \bar{p}_1^T(s) + \bar{p}_2(i) K_{\bar{v}}(i, s) \bar{p}_2^T(s)].
\end{aligned}$$

From (A-2), it is clear that



$$h(k, s, k) = \bar{\Phi}^k J_1(s, k). \quad (\text{A-35})$$

From (A-32), we have

$$\begin{aligned} q_1(k, k) &= P_1(k, k)(\bar{\Phi}^T)^k, \\ q_2(k, k) &= P_2(k, k)(\Phi_V^T)^k. \end{aligned} \quad (\text{A-36})$$

Putting $L = s$ in (A-27) and (A-28), we have.

$$\begin{aligned} P_1(k, k) &= \sum_{i=1}^L h(k, i, k) \bar{p}_1(i) \bar{H} \bar{B}(i), \\ P_2(k, k) &= \sum_{i=1}^L h(k, i, k) \bar{p}_2(i) B_V(i). \end{aligned} \quad (\text{A-37})$$

From (A-35)-(A-37) with (A-8), (A-9) and (A-18), we obtain $q_1(k, k) = S_{11}(k)$ and $q_2(k, k) = S_{12}(k)$.

From (A-29) and (A-32), we have

$$\begin{aligned} h(k, L, L) \bar{R}(s) &= K_{\bar{x}}(k, k)(\bar{\Phi}^T)^{L-k} \bar{H}^T \bar{p}_1^T(L) - q_1(k, L)(L) \bar{H}^T \bar{p}_1^T(L) \\ &\quad - q_2(k, L) \bar{p}_2^T(L). \end{aligned} \quad (\text{A-38})$$

Substituting (A-32) and (A-33) into (A-38), after some manipulations, we obtain (31).

Now, from (16), the filtering estimate $\hat{x}(L, L)$ is given by

$$\hat{x}(L, L) = \sum_{i=1}^L h(L, i, L) y(i).$$

Let us introduce functions.

$$\begin{aligned} e_1(L) &= \sum_{i=1}^L J_1(i, L) y(i), \\ e_2(L) &= \sum_{i=1}^L J_2(i, L) y(i). \end{aligned} \quad (\text{A-39})$$

From (A-35), we see that

$$\hat{x}(L, L) = \bar{\Phi}^L e_1(L). \quad (\text{A-40})$$

Subtracting the equation obtained by putting $L \rightarrow L-1$ in (A-39) from (A-39), we have

$$\begin{aligned} e_1(L) - e_1(L-1) &= J_1(L, L) y(L) + \sum_{i=1}^{L-1} (J_1(i, L) - J_1(i, L-1)) y(i), \\ e_1(0) &= 0. \end{aligned} \quad (\text{A-41})$$

Substituting (A-6) into (A-41), we obtain

$$e_1(L) = e_1(L-1) + J_1(L, L) y(L) - J_1(L, L) [\bar{p}_1(L) \bar{H} \bar{\Phi}^L e_1(L-1) + \bar{p}_2(L) \Phi_V^L e_2(L-1)]. \quad (\text{A-42})$$

Similarly, from (A-7), we obtain the difference equation for $e_2(L)$.

$$\begin{aligned} e_2(L) &= e_2(L-1) + J_2(L, L) y(L) - J_2(L, L) [\bar{p}_1(L) \bar{H} \bar{\Phi}^L e_1(L-1) + \bar{p}_2(L) \Phi_V^L e_2(L-1)], \\ e_2(0) &= 0. \end{aligned} \quad (\text{A-43})$$

Substituting (A-42) into (A-40), using (A-23) and introducing



$$\hat{\hat{v}}(L, L) = \Phi_V^L e_2(L), \quad (\text{A-44})$$

we have

$$\begin{aligned} \hat{\hat{x}}(L, L) &= \bar{\Phi} \hat{\hat{x}}(L-1, L-1) + G_1(L, L)[y(L) - (\bar{p}_1(L) \bar{H} \bar{\Phi} \hat{\hat{x}}(L-1, L-1) \\ &+ \bar{p}_2(L) \Phi_V \hat{\hat{v}}(L-1, L-1))], \hat{\hat{x}}(0, 0) = 0. \end{aligned} \quad (\text{A-45})$$

Similarly, the difference equation for $\hat{\hat{v}}(L, L)$ is obtained from (A-44), (A-23), (A-43) as (35).

The fixed-point smoothing estimate $\hat{\hat{x}}(k, L)$ is given by (16). Subtracting $\hat{\hat{x}}(k, L-1)$ from $\hat{\hat{x}}(k, L)$, we have

$$\hat{\hat{x}}(k, L) - \hat{\hat{x}}(k, L-1) = h(k, L, L)y(L) + \sum_{i=1}^{L-1} (h(k, i, L) - h(k, i, L-1))y(i). \quad (\text{A-46})$$

Substituting (A-4) into (A-46) and using (A-39), (A-40) and (A-44), we obtain

$$\begin{aligned} \hat{\hat{x}}(k, L) &= \hat{\hat{x}}(k, L-1) + h(k, L, L)(y(L) - \bar{p}_1(L) \bar{H} \bar{\Phi} \hat{\hat{x}}(L-1, L-1) \\ &- \bar{p}_2(L) \Phi_V \hat{\hat{v}}(L-1, L-1)). \end{aligned} \quad (\text{A-47})$$

Initial condition of the fixed-point smoothing estimate $\hat{\hat{x}}(k, L)$ at $L = k$ is the filtering estimate $\hat{\hat{x}}(k, k)$.

(Q.E.D.)

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