RLS Wiener Fixed-Point Smoother and Filter with Randomly Delayed or Uncertain Observations in Linear Discrete-Time Stochastic Descriptor Systems

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Abstract

The purpose of this paper is to design the recursive least-squares (RLS) Wiener fixed-point smoother and filter in linear discrete-time descriptor systems. The signal process is observed with additional observation noise. The observed value is randomly delayed by multiple sampling intervals or has the possibility of uncertainty that the observed value does not include the signal and contains the observation noise only. It is assumed that the probability of the observation delay and the probability that the observation does not contain the signal are known. The delayed or uncertain measurements are characterized by the Bernoulli random variables. The characteristic of this paper is that the RLS Wiener estimators are proposed from the randomly delayed, by multiple sampling intervals, or uncertain observations particularly for the descriptor systems in linear discrete-time stochastic systems.

Keywords: Discrete-time stochastic descriptor systems; RLS Wiener filter; RLS Wiener fixed-point smoother; randomly delayed observations; uncertain observations

1. Introduction

The descriptor systems have attracted attentions from the general expressions for physical models and structures in comparison with the conventional state-space models. The estimation problems for the descriptor systems have been studied ([1]-[8] and the references therein). Physical systems, e.g. a cart-pendulum system, electrical circuits [6], etc. are formulated as the descriptor systems. In [1], the recursive estimation algorithms for the filtering, prediction and fixed-lag smoothing estimates are presented, based on the innovation approach, for the descriptor systems with multiple packet dropouts and correlated noises. In [2], the information filter and predictor are proposed for the discrete-time descriptor systems with uncertain parameters. In addition to the nominal and robust estimation algorithms, the array algorithms are proposed. In [3], the robust Kalman type filter is proposed for the discrete-time descriptor systems with uncertainties in some matrices. In [4], according to the optimization technique, the optimum prediction estimate is updated in linear discrete-time descriptor systems. In [5], the robust predictor is presented for the discrete-time descriptor systems with bounded uncertainties. In [7], the problem of H_∞ filtering for descriptor systems with strict linear matrix inequalities (LMIs) is investigated. The necessary and sufficient conditions for the solvability and the expression of the solution are obtained for both continuous-time and discrete-time descriptor systems. In [8], the paper studies on the delay-dependent robust H_∞ filtering for uncertain discrete-time singular systems with the time-varying delay. Usually, the network data of control system are transmitted with delay and packet dropout from a sensor to a controller and also from a controller to an actuator [9]. In [10], for discrete-time stochastic linear systems with bounded random measurement delays and packet dropouts, the optimal filter, predictor and smoother are proposed based on the innovation approach. The observations are obtained in terms of (1c) in the paper. In [11], for the discrete-time stochastic systems with multiple packet dropouts, the optimal filter, predictor and smoother are proposed. Also, with observations multiply and randomly delayed, the recursive least-squares (RLS) Wiener fixed-point smoother and filter are proposed [12]. Under the condition that the uncertain observations are given, the RLS estimation algorithms are proposed, given the probability that the signal exists in the observation. The uncertain observation, if the signal exists, is characterized by using the independent Bernoulli random variable [13]. Estimation technique in [13] is extended to the case where the uncertain random variables are correlated [14]. In addition to the probability that the signal exists in the observation, the conditional probability is taken into account for the existence of the signal in the observation. In [15], the RLS Wiener fixed-point smoothing and filtering algorithms are proposed for the discrete-time stochastic system with uncertain observations. The probability that the signal exists in the observation and the conditional probability are required in the algorithms. With regard to multiple packet losses, related to the delayed observations, the optimal filter is devised in linear discrete-time stochastic systems over unreliable wireless sensor networks [16]. The technique is also extended to the extended Kalman filter in nonlinear discrete-time stochastic systems. In [17], the second order polynomial estimator is proposed in nonlinear systems with uncertain observations. The uncertainty in the observation equation is described by the Bernoulli random variables. In [18], the RLS Wiener fixed-point smoother and filter are proposed for the discrete-time stochastic systems with randomly delayed, by multiple sampling intervals, or uncertain observations. In [19], for the descriptor systems, based on the innovation theory, the RLS filter is proposed by using the covariance information of the state vector and the covariance information of the observation noise in linear discrete-time stochastic systems. Then the RLS Wiener type filter is presented for the descriptor systems.

In the packet dropout of the network systems, there might happen the case where the randomly delayed observed value does not include the signal. The packet dropout might not correspond to the certain observation including the signal, and also to the delayed observation. By the uncertain observation it means that there exists uncertainty if the observed value includes the signal or not [13]. In the packet dropout, the observation not including the signal data might further delay by one or more sampling intervals. As described above, the estimation problems for the descriptor systems have drawn great interests from the nature of the generalized state-space model. From this viewpoint, this paper, based on the estimation techniques both in [18] and [19], examines to design the RLS Wiener fixed-point smoother and filter from randomly delayed observed values, by multiple sampling intervals, or uncertain observations in linear discrete-time descriptor systems. The signal process is observed with additional observation noise. The observed value is randomly delayed by multiple sampling intervals or has the possibility of uncertainty that the observed value does not include the signal and contains the observation noise only. It is assumed that the probability of the random observation delay and the probability that the observation does not contain the signal are known as a priori information. The randomly delayed or uncertain measurements are characterized by the Bernoulli random variables. The characteristic of this paper is that the RLS Wiener estimators are proposed from the randomly delayed, by multiple sampling intervals, or uncertain observations particularly for the descriptor systems in linear discrete-time stochastic systems.

A numerical simulation example, in section 5, shows the estimation characteristics of the proposed fixed-point smoother and filter with the randomly delayed, by multiple sampling intervals, or uncertain observations in linear discrete-time descriptor systems.

2. Least-squares smoothing problem with delayed or uncertain observations for descriptor systems

In linear discrete-time descriptor systems, the state and observation equations are described by [1]

$$\begin{aligned} &\Xi S(k+1) = FS(k) + \Lambda w(k), E[w(k)w^T(s)] = Q\delta_K(k-s), \\ &\bar{y}(k) = CS(k) + v(k), E[v(k)v^T(s)] = R\delta_K(k-s). \end{aligned} \quad (1)$$

Here, $S(k)$ represents an n-dimensional descriptor vector, $w(k)$ a q-dimensional input noise and $\bar{y}(k)$ an m-dimensional certain measurement without including delays or uncertain signals. Ξ, F, Λ and C are matrices with the dimensions $n\times n$, $n\times n$, $n\times q$, $m\times n$ respectively. In the descriptor systems, Ξ is the singular matrix, i.e. $rank(\Xi) < n$. In terms of orthogonal matrices U and V, the singular value decomposition (SVD) of Ξ is written as follows.

$$\Xi = UDV^T, D = \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix}, U^T = U^{-1}, V^T = V^{-1}, \\ \Delta = diag(\mu_1, \mu_2, ..., \mu_l), \mu_i > 0, i = 1,2,...,l, \Delta > 0 \quad (2)$$

From the relationships

$$U^T FV = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix},\ U^T \Lambda = \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix},\ CV = \begin{bmatrix} C_1 & C_2 \end{bmatrix},\ S(k) = V \begin{bmatrix} \bar{S}_1(k) \\ \bar{S}_2(k) \end{bmatrix},$$

the state equation in (1) is described by

$$\begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{S}_1(k+1) \\ \bar{S}_2(k+1) \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} \bar{S}_1(k) \\ \bar{S}_2(k) \end{bmatrix} + \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix} w(k). \quad (3)$$

From (3), it follows that

$$\bar{S}_1(k+1) = A\bar{S}_1(k) + Bw(k), A = \Delta^{-1}(F_{11} + F_{12}\Gamma_1), B = \Delta^{-1}(F_{12}\Gamma_2 + \Lambda_1), \Delta > 0 \\ \bar{y}(k) = H\bar{S}_1(k) + \bar{v}(k), H = C_1 + C_2\Gamma_1, \bar{v}(k) = C_2\Gamma_2 w(k) + v(k), \\ E[\bar{v}(k)\bar{v}^T(s)] = \bar{R}\delta_K(k-s), \bar{R} = C_2\Gamma_2 Q\Gamma_2^T C_2^T + R, \\ \bar{S}_2(k) = \Gamma_1\bar{S}_1(k) + \Gamma_2 w(k), \Gamma_1 = -F_{22}^{-1}F_{21}, \Gamma_2 = -F_{22}^{-1}\Lambda_2, F_{22} > 0. \quad (4)$$

For notational conveniences, let us put $\bar{S}_1(k)$ as $x(k) = \bar{S}_1(k)$ and A as $\Phi = A$. Then the state equation for the state vector $x(k)$ and its observation equation are given by

$$x(k+1) = \Phi x(k) + Bw(k), E[w(k)w^T(s)] = Q\delta_K(k-s), \\ \bar{y}(k) = z(k) + \bar{v}(k), z(k) = Hx(k), E[\bar{v}(k)\bar{v}^T(s)] = \bar{R}\delta_K(k-s). \quad (5)$$

For the discrete-time systems with measurement delays or uncertain observations, let an m-dimensional observation equation be described as

$$y(k) = \gamma_{01}(k)\bar{y}(k) + \gamma_{11}(k)\bar{y}(k-1) + \cdots + \gamma_{\bar{N}1}(k)\bar{y}(k-\bar{N}) + \gamma_{00}(k)\bar{v}(k) + \gamma_{10}(k)\bar{v}(k-1) + \cdots \\ + \gamma_{\bar{N}0}(k)\bar{v}(k-\bar{N}), \\ \bar{y}(k) = z(k) + \bar{v}(k), z(k) = Hx(k), \\ E[\gamma_{01}(k)] = p_{01}(k), E[\gamma_{11}(k)] = p_{11}(k), E[\gamma_{21}(k)] = p_{21}(k), \cdots, E[\gamma_{\bar{N}1}(k)] = p_{\bar{N}1}(k), \\ E[\gamma_{00}(k)] = p_{00}(k), E[\gamma_{10}(k)] = p_{10}(k), E[\gamma_{20}(k)] = p_{20}(k), \cdots, E[\gamma_{\bar{N}0}(k)] = p_{\bar{N}0}(k). \quad (6)$$

Let us assume that the observation at each time $k > 1$ can either be delayed by sampling intervals j, $1 \le j \le \bar{N}$, with known probabilities or consists of delayed measurements, which do not contain signal data. $\{\gamma_{ij}(k), 0 \le i \le \bar{N}, j = 0,1; \quad k > 1\}$ represent a sequence of Bernoulli random variables (binary switching sequence taking the values 0 or 1 with $P[\gamma_{ij}(k) = 1] = p_{ij}(k), 0 \le i \le \bar{N}, j = 0,1$). By introducing the notations

$$\begin{aligned}
&\bar{\gamma}_1(k) = \begin{bmatrix} \gamma_{01}(k) I_{m\times m} & \gamma_{11}(k) I_{m\times m} & \gamma_{21}(k) I_{m\times m} & \cdots & \gamma_{\bar{N}1}(k) I_{m\times m} \end{bmatrix}, \\
&\bar{\gamma}_0(k) = \begin{bmatrix} \gamma_{00}(k) I_{m\times m} & \gamma_{10}(k) I_{m\times m} & \gamma_{20}(k) I_{m\times m} & \cdots & \gamma_{\bar{N}0}(k) I_{m\times m} \end{bmatrix}, \\
&\tilde{y}(k) = \begin{bmatrix} \bar{y}(k) & \bar{y}(k-1) & \cdots & \bar{y}(k-\bar{N}) \end{bmatrix}^T, \\
&\tilde{v}(k) = \begin{bmatrix} \bar{v}(k) & \bar{v}(k-1) & \cdots & \bar{v}(k-\bar{N}) \end{bmatrix}^T,
\end{aligned} \tag{7}$$

from (6), we obtain

$$y(k) = \bar{\gamma}_1(k)\tilde{y}(k) + \bar{\gamma}_0(k)\tilde{v}(k)\,. \tag{8}$$

$\bar{\gamma}_1(k)$ corresponds to the Bernoulli random variables for the measurement delays and $\bar{\gamma}_0(k)$ for the observations, which consist of only observation noise data. Let $E_\gamma[\cdot]$ represents the expectation with respect to the random variables $\{\gamma(k),\quad k \ge 1\}$. The Bernoulli random variables satisfy $E_\gamma[\gamma_{ij}(k)] = p_{ij}(k) I_{m\times m}$, $E_\gamma[\gamma_{ij}^2(k)] = p_{ij}(k) I_{m\times m}$, $0 \le i \le \bar{N}, j = 0,1$. It is found that the auto-covariance function $K_{\tilde{V}}(k,s)$ of $\tilde{v}(k)$ is given by

$$K_{\tilde{V}}(k,s) = \begin{cases} \bar{C}(k)\bar{D}^T(s), 0 \le s \le k, \\ \bar{D}(k)\bar{C}^T(s), 0 \le k \le s, \end{cases} \tag{9}$$

$\bar{C}(k) = \Phi_{\tilde{V}}^k, \bar{D}^T(s) = \Phi_{\tilde{V}}^{-s} K_{\tilde{V}}(s,s)$. Here, the transition matrix $\Phi_{\tilde{V}}$ and the variance $K_{\tilde{V}}(s,s)$ of $\tilde{v}(k)$ are given as follows.

$$\Phi_{\tilde{V}} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ I_{m\times m} & 0 & \cdots & 0 & 0 \\ 0 & I_{m\times m} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_{m\times m} & 0 \end{bmatrix}, K_{\tilde{V}}(s,s) = \begin{bmatrix} \bar{R} & 0 & \cdots & 0 & 0 \\ 0 & \bar{R} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \bar{R} & 0 \\ 0 & 0 & \cdots & 0 & \bar{R} \end{bmatrix} \tag{10}$$

By introducing

$$\bar{H} = \begin{bmatrix} H & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & H & 0 \\ 0 & \cdots & 0 & H \end{bmatrix},\ \bar{x}(k) = \begin{bmatrix} x(k) \\ x(k-1) \\ \vdots \\ x(k-\bar{N}) \end{bmatrix}, \tag{11}$$

from (6) and (7), the observation equation (8) is rewritten as

$$y(k) = \bar{\gamma}_1(k)\bar{H}\bar{x}(k) + \bar{\gamma}_1(k)\tilde{v}(k) + \bar{\gamma}_0(k)\tilde{v}(k)\,, \tag{12}$$

since

$$\bar{z}(k) = \begin{bmatrix} z(k) \\ z(k-1) \\ \vdots \\ z(k-\bar{N}) \end{bmatrix} = \begin{bmatrix} Hx(k) \\ Hx(k-1) \\ \vdots \\ Hx(k-\bar{N}) \end{bmatrix} = \bar{H}\bar{x}(k)\,. \tag{13}$$

Let $K_x(k,s)=K_x(k-s)$ denote the auto-covariance function of the state vector $x(k)$ in wide-sense stationary stochastic systems [20], and let $K_x(k,s)$ be expressed by

$$K_x(k,s)=\begin{cases} A(k)B^T(s), & 0\le s\le k, \\ B(s)A^T(k), & 0\le k\le s, \end{cases} \tag{14}$$

$A(k)=\Phi^k$, $B^T(s)=\Phi^{-s}K_x(s,s)$. Here, Φ represents the transition matrix of $x(k)$. From

$$\begin{bmatrix} x(k+1) \\ x(k) \\ \vdots \\ x(k-\bar{N}+2) \\ x(k-\bar{N}+1) \end{bmatrix}=\begin{bmatrix} \Phi & 0 & \cdots & 0 & 0 \\ I_{n\times n} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & I_{n\times n} & 0 \end{bmatrix}\begin{bmatrix} x(k) \\ x(k-1) \\ \vdots \\ x(k-\bar{N}+1) \\ x(k-\bar{N}) \end{bmatrix}+\begin{bmatrix} w(k) \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \tag{15}$$

The system matrix $\bar{\Phi}$ for the state vector $\bar{x}(k)$ is given by

$$\bar{\Phi}=\begin{bmatrix} \Phi & 0 & \cdots & 0 & 0 \\ I_{n\times n} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & I_{n\times n} & 0 \end{bmatrix}. \tag{16}$$

Let $K_{\bar{x}}(k,s)$ represent the auto-covariance function of $\bar{x}(k)$. Then $K_{\bar{x}}(k,s)$ is given by

$$K_{\bar{x}}(k,s)=\begin{cases} \bar{A}(k)\bar{B}^T(s), 0\le s\le k, \\ \bar{B}(k)\bar{A}^T(s), 0\le k\le s, \end{cases} \tag{17}$$

$\bar{A}(k)=\bar{\Phi}^k$, $\bar{B}^T(s)=\bar{\Phi}^{-s}K_{\bar{x}}(s,s)$. Here, $K_{\bar{x}}(s,s)=K_{\bar{x}}(0)$ is described as

$$K_{\bar{x}}(s,s)=E\left[\begin{bmatrix} x(s) \\ x(s-1) \\ \vdots \\ x(s-\bar{N}+1) \\ x(s-\bar{N}) \end{bmatrix}\begin{bmatrix} x^T(s) & x^T(s-1) & \cdots & x^T(s-\bar{N}+1) & x^T(s-\bar{N}) \end{bmatrix}\right]$$
$$=\begin{bmatrix} K_x(0) & \Phi K_x(0) & \cdots & \Phi^{\bar{N}-1}K_x(0) & \Phi^{\bar{N}}K_x(0) \\ K_x(0)\Phi^T & K_x(0) & \cdots & \Phi^{\bar{N}-2}K_x(0) & \Phi^{\bar{N}-1}K_x(0) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ K_x(0)(\Phi^T)^{\bar{N}-1} & K_x(0)(\Phi^T)^{\bar{N}-2} & \cdots & K_x(0) & \Phi K_x(0) \\ K_x(0)(\Phi^T)^{\bar{N}} & K_x(0)(\Phi^T)^{\bar{N}-1} & \cdots & K_x(0)\Phi^T & K_x(0) \end{bmatrix}. \tag{18}$$

Let the fixed-point smoothing estimate $\hat{\bar{x}}(k,L)$ of $\bar{x}(k)$ at the fixed point k be given by

$$\hat{\bar{x}}(k,L)=\sum_{i=1}^{L}h(k,i,L)y(i) \tag{19}$$

in terms of the observed values $\{y(i),\quad 1\le i\le L\}$. In (19), $h(k,i,L)$ is a time-varying impulse response function. We consider the estimation problem, which minimizes the mean-square value (MSV)

$$J=E[\|\bar{x}(k)-\hat{\bar{x}}(k,L)\|^2] \tag{20}$$

of the fixed-point smoothing error. Based on an orthogonal projection lemma [20],

$$\bar{x}(k)-\sum_{i=1}^{L}h(k,i,L)y(i)\perp y(s),\ 1\le s\le L, \tag{21}$$

the optimal impulse response function satisfies the Wiener-Hopf equation

$$E[\bar{x}(k)y^T(s)]=\sum_{i=1}^{L}h(k,i,L)E[y(i)y^T(s)]. \tag{22}$$

Here '$\perp$' denotes the notation of the orthogonality. By introducing $\bar{p}_1(k)=\left[p_{01}(k)I_{m\times m}\quad p_{11}(k)I_{m\times m}\quad \cdots\quad p_{\bar{N}-11}(k)I_{m\times m}\quad p_{\bar{N}1}(k)I_{m\times m}\right]$, from (7) and (12), the left hand side of (22) is developed as

$$E[\bar{x}(k)y^T(s)]=E[\bar{x}(k)(\bar{\gamma}_1(s)\bar{H}\bar{x}(s)+\bar{\gamma}_1(s)\tilde{v}(s)+\bar{\gamma}_0(s)\tilde{v}(s))^T]$$

$$=E[\bar{x}(k)\bar{x}^T(s)]\bar{H}^T\left[p_{01}(s)I_{m\times m}\quad p_{11}(s)I_{m\times m}\quad\cdots\quad p_{\bar{N}-11}(s)I_{m\times m}\quad p_{\bar{N}1}(s)I_{m\times m}\right]^T$$

$$=K_{\bar{x}}(k,s)\bar{H}^T\bar{p}_1^T(s). \tag{23}$$

Also, $E[y(i)y^T(s)]$ is obtained as

$$E[y(i)y^T(s)]=E[(\bar{\gamma}_1(i)\bar{H}\bar{x}(i)+\bar{\gamma}_1(i)\tilde{v}(i)+\bar{\gamma}_0(i)\tilde{v}(i))(\bar{\gamma}_1(s)\bar{H}\bar{x}(s)+\bar{\gamma}_1(s)\tilde{v}(s)+\bar{\gamma}_0(s)\tilde{v}(s))^T]$$
$$=E_\gamma[(\bar{\gamma}_1(i)\bar{H}K_{\bar{x}}(i,s)\bar{H}^T\bar{\gamma}_1^T(s)]+E_\gamma[\bar{\gamma}_2(i)K_{\tilde{V}}(i,s)\bar{\gamma}_2^T(s)],$$

$$\bar{\gamma}_2(s)=\bar{\gamma}_0(s)+\bar{\gamma}_1(s). \tag{24}$$

Substituting (23) and (24) into (22), we have

$$K_{\bar{x}}(k,s)H^T\bar{p}_1^T(s)=\sum_{i=1}^{L}h(k,i,L)\{E_\gamma[\bar{\gamma}_1(i)\bar{H}K_{\bar{x}}(i,s)\bar{H}^T\bar{\gamma}_1^T(s)]+E_\gamma[\bar{\gamma}_2(i)K_{\tilde{V}}(i,s)\bar{\gamma}_2^T(s)]\}. \tag{25}$$

From the stochastic property of $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$, (25) is rewritten as

$$K_{\bar{x}}(k,s)H^T\bar{p}_1^T(s)=h(k,s,L)\{E_\gamma[\bar{\gamma}_1(s)\bar{H}K_{\bar{x}}(s,s)\bar{H}^T\bar{\gamma}_1^T(s)]+E_\gamma[\bar{\gamma}_2(s)K_{\tilde{V}}(s,s)\bar{\gamma}_2^T(s)]$$
$$-E_\gamma[\bar{\gamma}_1(s)]\bar{H}K_{\bar{x}}(s,s)\bar{H}^TE_\gamma[\bar{\gamma}_1^T(s)]-E_\gamma[\bar{\gamma}_2(s)]K_{\tilde{V}}(s,s)E_\gamma[\bar{\gamma}_2^T(s)]\} \quad (26)$$
$$+\sum_{i=1}^{L}h(k,i,L)\{E_\gamma[\bar{\gamma}_1(i)]\bar{H}K_{\bar{x}}(i,s)\bar{H}^TE_\gamma[\bar{\gamma}_1^T(s)]+E_\gamma[\bar{\gamma}_2(i)]K_{\tilde{V}}(i,s)E_\gamma[\bar{\gamma}_2^T(s)]\}.$$

Rearranging (26) and introducing $\breve{R}(s)$, we obtain the equation for the optimal impulse response function $h(k,s,L)$ as

$$h(k,s,L)\breve{R}(s)=K_{\bar{x}}(k,s)\bar{H}^T\bar{p}_1^T(s)$$
$$-\sum_{i=1}^{L}h(k,i,L)\{\bar{p}_1(i)\bar{H}K_{\bar{x}}(i,s)\bar{H}^T\bar{p}_1^T(s)+\bar{p}_2(i)K_{\tilde{V}}(i,s)\bar{p}_2^T(s)\}, \quad (27)$$

$$\breve{R}(s)=E_\gamma[\bar{\gamma}_1(s)\bar{H}K_{\bar{x}}(s,s)\bar{H}^T\bar{\gamma}_1^T(s)]+E_\gamma[\bar{\gamma}_2(s)K_{\tilde{V}}(s,s)\bar{\gamma}_2^T(s)]$$
$$-E_\gamma[\bar{\gamma}_1(s)]\bar{H}K_{\bar{x}}(s,s)\bar{H}^TE_\gamma[\bar{\gamma}_1^T(s)]-E_\gamma[\bar{\gamma}_2(s)]K_{\tilde{V}}(s,s)E_\gamma[\bar{\gamma}_2^T(s)]. \quad (28)$$

3. RLS Wiener estimation algorithms with delayed or uncertain measurements for descriptor systems

Under the problem formulation in section 2 on the linear least-squares estimation for the descriptor systems with the randomly delayed, by multiple sampling intervals, or uncertain observations , Theorem 1 presents the RLS Wiener fixed-point smoothing and filtering algorithms of the descriptor vector $S(k)$.

Theorem 1

Based on the optimal estimation problems in section 2, the RLS Wiener algorithms for the fixed-point smoothing and filtering estimates of the descriptor vector $S(k)$ consist of (29)-(47) for the descriptor systems with randomly delayed, by multiple sampling intervals, or uncertain observations in linear discrete-time stochastic systems.

Fixed-point smoothing estimate of the descriptor vector $S(k)=V\begin{bmatrix}\bar{S}_1(k)\\ \bar{S}_2(k)\end{bmatrix}$: $\hat{S}(k,L)$

$$\hat{S}(k,L)=V\begin{bmatrix}\hat{\bar{S}}_1(k,L)\\ \hat{\bar{S}}_2(k,L)\end{bmatrix} \quad (29)$$

Fixed-point smoothing estimate of $\bar{S}_1(k)$ at the fixed point k : $\hat{\bar{S}}_1(k,L)$

$$\hat{\bar{S}}_1(k,L)=\begin{bmatrix}I_{l\times l} & 0_{l\times l\cdot N}\end{bmatrix}\hat{\bar{x}}(k,L) \quad (30)$$

Fixed-point smoothing estimate of $\bar{S}_2(k)$: $\hat{\bar{S}}_2(k,L)$

$$\hat{\bar{S}}_2(k,L)=\Gamma_1\hat{\bar{S}}_1(k,L),$$
$$\Gamma_1=-F_{22}^{-1}F_{21},\Xi=UDV^T,D=\begin{bmatrix}\Delta & 0\\ 0 & 0\end{bmatrix},U^T=U^{-1},V^T=V^{-1}, \quad (31)$$
$$\Delta=diag(\mu_1,\mu_2,...,\mu_l),\mu_i>0,i=1,2,...,l,\Delta>0,U^TFV=\begin{bmatrix}F_{11} & F_{12}\\ F_{21} & F_{22}\end{bmatrix}$$

Filtering estimate of the descriptor vector $S(k)=V\begin{bmatrix}\bar{S}_1(k)\\ \bar{S}_2(k)\end{bmatrix}$: $\hat{S}(k,k)$

$$\hat{S}(k,k)=V\begin{bmatrix}\hat{\bar{S}}_1(k,k)\\ \hat{\bar{S}}_2(k,k)\end{bmatrix} \quad (32)$$

Filtering estimate of $\bar{S}_1(k)$: $\hat{\bar{S}}_1(k,k)$

$$\hat{\bar{S}}_1(k,k)=\begin{bmatrix}I_{l\times l} & 0_{l\times l\cdot\bar{N}}\end{bmatrix}\hat{\bar{x}}(k,k) \quad (33)$$

Filtering estimate of $\bar{S}_2(k)$: $\hat{\bar{S}}_2(k,k)$

$$\hat{\bar{S}}_2(k,k)=\Gamma_1\hat{\bar{S}}_1(k,k) \quad (34)$$

Fixed-point smoothing estimate of $\bar{x}(k)$ at the fixed point k: $\hat{\bar{x}}(k,L)$

$$\hat{\bar{x}}(k,L)=\hat{\bar{x}}(k,L-1)+h(k,L,L)(y(L)-\bar{p}_1(L)\bar{H}\bar{\Phi}\hat{\bar{x}}(L-1,L-1)-\bar{p}_2(L)\Phi_{\breve{V}}\hat{\breve{v}}(L-1,L-1)),$$

$$\hat{\bar{x}}(k,L)=\begin{bmatrix}\hat{x}(k,L)\\ \hat{x}(k-1,L)\\ \vdots\\ \hat{x}(k-\bar{N},L)\end{bmatrix} \quad (35)$$

Smoother gain: $h(k,L,L)$

$$\begin{aligned}h(k,L,L)=&[K_{\bar{x}}(k,k)(\bar{\Phi}^T)^{L-k}\bar{H}^T\bar{p}_1^T(L)-q_1(k,L-1)\bar{\Phi}^T\bar{H}^T\bar{p}_1^T(L)-q_2(k,L-1)\Phi_{\breve{V}}^T\bar{p}_2^T(L)]\\ &\times\{\breve{R}(L)+[\bar{p}_1(L)\bar{H}K_{\bar{x}}(L,L)-\bar{p}_1(L)\bar{H}\bar{\Phi}S_{11}(L-1)\bar{\Phi}^T-\bar{p}_2(L)\Phi_{\breve{V}}S_{21}(L-1)\bar{\Phi}^T]\bar{H}^T\bar{p}_1^T(L)\\ &+[\bar{p}_2(L)K_{\breve{V}}(L,L)-\bar{p}_1(L)\bar{H}\bar{\Phi}S_{12}(L-1)\Phi_{\breve{V}}^T-\bar{p}_2(L)\Phi_{\breve{V}}S_{22}(L-1)\Phi_{\breve{V}}^T]\bar{H}^T\bar{p}_2^T(L)\}^{-1}\end{aligned} \quad (36)$$

$$\begin{aligned}q_1(k,L)=&q_1(k,L-1)\bar{\Phi}^T+h(k,L,L)[\bar{p}_1(L)\bar{H}K_{\bar{x}}(L,L)-\bar{p}_1(L)\bar{H}\bar{\Phi}S_{11}(L-1)\bar{\Phi}^T\\ &-\bar{p}_2(L)\Phi_{\breve{V}}S_{21}(L-1)\bar{\Phi}^T],\end{aligned} \quad (37)$$

$$q_1(k,k)=S_{11}(k)$$

$$q_2(k,L)=q_2(k,L-1)\Phi_{\tilde{V}}^T+h(k,L,L)[\bar{p}_2(L)K_{\tilde{V}}(L,L)-\bar{p}_1(L)\bar{H}\bar{\Phi}S_{12}(L-1)\Phi_{\tilde{V}}^T$$
$$-\bar{p}_2(L)\Phi_{\tilde{V}}S_{22}(L-1)\Phi_{\tilde{V}}^T], \quad (38)$$

$$q_2(k,k)=S_{12}(k)$$

Filtering estimate of $\bar{x}(L)$: $\hat{\bar{x}}(L,L)$

$$\hat{\bar{x}}(L,L)=\bar{\Phi}\hat{\bar{x}}(L-1,L-1)+G_1(L,L)(y(L)-\bar{p}_1(L)\bar{H}\bar{\Phi}\hat{\bar{x}}(L-1,L-1)$$
$$-\bar{p}_2(L)\Phi_{\tilde{V}}\hat{\tilde{v}}(L-1,L-1)), \quad (39)$$

$$\hat{\bar{x}}(0,0)=0$$

Filtering estimate of $\tilde{v}(L)$: $\hat{\tilde{v}}(L,L)$

$$\hat{\tilde{v}}(L,L)=\Phi_{\tilde{V}}\hat{\tilde{v}}(L-1,L-1)+G_2(L,L)(y(L)-\bar{p}_1(L)\bar{H}\bar{\Phi}\hat{\bar{x}}(L-1,L-1)$$
$$-\bar{p}_2(L)\Phi_{\tilde{V}}\hat{\tilde{v}}(L-1,L-1)), \quad (40)$$
$$\hat{\tilde{v}}(0,0)=0$$

Auto-variance function of $\hat{\bar{x}}(L,L)$: $S_{11}(L)=E[\hat{\bar{x}}(L,L)\hat{\bar{x}}^T(L,L)]$

$$S_{11}(L)=\bar{\Phi}S_{11}(L-1)\bar{\Phi}^T+G_1(L,L)[\bar{p}_1(L)\bar{H}K_{\bar{x}}(L,L)-\bar{p}_1(L)\bar{H}\bar{\Phi}S_{11}(L-1)\bar{\Phi}^T$$
$$-\bar{p}_2(L)\Phi_{\tilde{V}}S_{21}(L-1)\bar{\Phi}^T], \quad (41)$$
$$S_{11}(0)=0$$

Cross-variance function of $\hat{\bar{x}}(L,L)$ with $\hat{\tilde{v}}(L,L)$: $S_{12}(L)=E[\hat{\bar{x}}(L,L)\hat{\tilde{v}}^T(L,L)]$

$$S_{12}(L)=\bar{\Phi}S_{12}(L-1)\Phi_{\tilde{V}}^T+G_1(L,L)[\bar{p}_2(L)K_{\tilde{V}}(L,L)-\bar{p}_1(L)\bar{H}\bar{\Phi}S_{12}(L-1)\Phi_{\tilde{V}}^T$$
$$-\bar{p}_2(L)\Phi_{\tilde{V}}S_{22}(L-1)\Phi_{\tilde{V}}^T], \quad (42)$$
$$S_{12}(0)=0$$

$$S_{21}(L)=\Phi_{\tilde{V}}S_{21}(L-1)\bar{\Phi}^T+G_2(L,L)[\bar{p}_1(L)\bar{H}K_{\bar{x}}(L,L)-\bar{p}_1(L)\bar{H}\bar{\Phi}S_{11}(L-1)\bar{\Phi}^T$$
$$-\bar{p}_2(L)\Phi_{\tilde{V}}S_{21}(L-1)\bar{\Phi}^T], \quad (43)$$
$$S_{21}(0)=0, S_{21}(L)=S_{12}^T(L)$$

Auto-variance function of $\hat{\tilde{v}}(L,L)$: $S_{22}(L)=E[\hat{\tilde{v}}(L,L)\hat{\tilde{v}}^T(L,L)]$

$$S_{22}(L)=\Phi_{\tilde{V}}S_{22}(L-1)\Phi_{\tilde{V}}^T+G_2(L,L)[\bar{p}_2(L)K_{\tilde{V}}(L,L)-\bar{p}_1(L)\bar{H}\bar{\Phi}S_{12}(L-1)\Phi_{\tilde{V}}^T$$
$$-\bar{p}_2(L)\Phi_{\tilde{V}}S_{22}(L-1)\Phi_{\tilde{V}}^T], \quad (44)$$
$$S_{22}(0)=0$$

$$\begin{aligned}&G_1(L,L)=[K_{\bar{x}}(L,L)\bar{H}^T\bar{p}_1^T(L)-\bar{\Phi}S_{11}(L-1)\bar{\Phi}^T\bar{H}^T\bar{p}_1^T(L)-\bar{\Phi}S_{12}(L-1)\Phi_{\bar{V}}^T\bar{p}_2^T(L)]\\&\times\{\breve{R}(L)+[\bar{p}_1(L)\bar{H}K_{\bar{x}}(L,L)-\bar{p}_1(L)\bar{H}\bar{\Phi}S_{11}(L-1)\bar{\Phi}^T-\bar{p}_2(L)\Phi_{\bar{V}}S_{21}(L-1)\bar{\Phi}^T]\bar{H}^T\bar{p}_1^T(L)\\&+[\bar{p}_2(L)K_{\bar{V}}(L,L)-\bar{p}_1(L)\bar{H}\bar{\Phi}S_{12}(L-1)\Phi_{\bar{V}}^T-\bar{p}_2(L)\Phi_{\bar{V}}S_{22}(L-1)\Phi_{\bar{V}}^T]\bar{H}^T\bar{p}_2^T(L)\}^{-1}\end{aligned} \tag{45}$$

$$\begin{aligned}&G_2(L,L)=[K_{\bar{V}}(L,L)\bar{p}_2^T(L)-\Phi_{\bar{V}}S_{21}(L-1)\bar{\Phi}^T\bar{H}^T\bar{p}_1^T(L)-\Phi_{\bar{V}}S_{22}(L-1)\Phi_{\bar{V}}^T\bar{p}_2^T(L)]\\&\times\{\breve{R}(L)+[\bar{p}_1(L)\bar{H}K_{\bar{x}}(L,L)-\bar{p}_1(L)\bar{H}\bar{\Phi}S_{11}(L-1)\bar{\Phi}^T-\bar{p}_2(L)\Phi_{\bar{V}}S_{21}(L-1)\bar{\Phi}^T]\bar{H}^T\bar{p}_1^T(L)\\&+[\bar{p}_2(L)K_{\bar{V}}(L,L)-\bar{p}_1(L)\bar{H}\bar{\Phi}S_{12}(L-1)\Phi_{\bar{V}}^T-\bar{p}_2(L)\Phi_{\bar{V}}S_{22}(L-1)\Phi_{\bar{V}}^T]\bar{H}^T\bar{p}_2^T(L)\}^{-1}\end{aligned} \tag{46}$$

$$\begin{aligned}&\breve{R}(L)=E_\gamma[\bar{\gamma}_1(L)\bar{H}K_{\bar{x}}(L,L)\bar{H}^T\bar{\gamma}_1^T(L)]+E_\gamma[\bar{\gamma}_2(L)K_{\bar{V}}(L,L)\bar{\gamma}_2^T(L)]\\&-E_\gamma[\bar{\gamma}_1(L)]\bar{H}K_{\bar{x}}(L,L)\bar{H}^TE_\gamma[\bar{\gamma}_1^T(L)]-E_\gamma[\bar{\gamma}_2(L)]K_{\bar{V}}(L,L)E_\gamma[\bar{\gamma}_2^T(L)]\end{aligned} \tag{47}$$

Theorem 1 is proved by referring to the RLS Wiener filter and fixed-point smoother for the systems with randomly delayed, by multiple sampling intervals, or uncertain observations [18] and the RLS Wiener filter for the descriptor systems [19] in linear discrete-time stochastic systems.

As the condition for the asymptotic stability of the filtering equation in Theorem 1 for $\hat{\bar{x}}(L,L)$, it is necessary that all the eigenvalues of $\bar{\Phi}-G_1(L,L)\bar{p}_1(L)\bar{H}\bar{\Phi}$ lie inside the unit circle. In addition, for the stability of the estimation algorithms, from (36), (45) and (46), the following matrix must be positive definite.

$$\begin{aligned}&\breve{R}(L)+[\bar{p}_1(L)\bar{H}K_{\bar{x}}(L,L)-\bar{p}_1(L)\bar{H}\bar{\Phi}S_{11}(L-1)\bar{\Phi}^T-\bar{p}_2(L)\Phi_{\bar{V}}S_{21}(L-1)\bar{\Phi}^T]\bar{H}^T\bar{p}_1^T(L)\\&+[\bar{p}_2(L)K_{\bar{V}}(L,L)-\bar{p}_1(L)\bar{H}\bar{\Phi}S_{12}(L-1)\Phi_{\bar{V}}^T-\bar{p}_2(L)\Phi_{\bar{V}}S_{22}(L-1)\Phi_{\bar{V}}^T]\bar{H}^T\bar{p}_2^T(L)>0\end{aligned}$$

4. A numerical simulation example

Let a scalar observation equation be given by

$$\bar{y}(k)=CS(k)+v(k), E[v(k)v(s)]=R\delta_K(k-s), \tag{48}$$

for the descriptor system

$$\begin{aligned}&\Xi S(k+1)=FS(k)+\Lambda w(k), E[w(k)w(s)]=Q\delta_K(k-s),\\&S(k)=\begin{bmatrix}S_1(k)\\S_2(k)\\S_3(k)\end{bmatrix},\Xi=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix},F=\begin{bmatrix}0&1&1\\-0.8&-0.1&0.5\\0&0.5&1.5\end{bmatrix},\Lambda=\begin{bmatrix}0.5\\1\\0.2\end{bmatrix},Q=0.5^2,C=\begin{bmatrix}0.5&0.9&0.6\end{bmatrix},\end{aligned} \tag{49}$$

in linear discrete-time stochastic systems. Here, $S(k)$ is the descriptor vector, $w(k)$ denotes the input noise and $\bar{y}(k)$ is the certain measurement without considering measurement delays or uncertain signals. Here, Ξ is the singular matrix, i.e. $rank(\Xi)=2<3$. With orthogonal matrices U and V, the SVD of Ξ is expressed by

$$\Xi=UDV^T, D=\begin{bmatrix}\Delta&0\\0&0\end{bmatrix}, U=V=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\Delta=\begin{bmatrix}1&0\\0&1\end{bmatrix}. \tag{50}$$

From the relationships

$$U^T F V = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix},\ U^T \Lambda = \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix},\ CV = \begin{bmatrix} C_1 & C_2 \end{bmatrix},\ S(k) = \begin{bmatrix} S_1(k) \\ S_2(k) \\ S_3(k) \end{bmatrix} = V \begin{bmatrix} \bar{S}_1(k) \\ \bar{S}_2(k) \end{bmatrix},$$

the state equation in (49) and the observation equation in (48),

$$\begin{aligned}
&\begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \bar{S}_1(k+1) \\ \bar{S}_2(k+1) \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}\begin{bmatrix} \bar{S}_1(k) \\ \bar{S}_2(k) \end{bmatrix} + \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix} w(k), \\
&\bar{y}(k) = CS(k) + v(k), E[v(k)v(s)] = R\delta_K(k-s), \\
&F_{11} = \begin{bmatrix} 0 & 1 \\ -0.8 & -0.1 \end{bmatrix}, F_{12} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, F_{21} = \begin{bmatrix} 0 & 0.5 \end{bmatrix}, F_{22} = 1.5, \\
&\Lambda_1 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \Lambda_2 = 0.2, \\
&C_1 = \begin{bmatrix} 0.5 & 0.9 \end{bmatrix}, C_2 = 0.6,
\end{aligned} \tag{51}$$

are transformed into

$$\begin{aligned}
&\bar{S}_1(k+1) = A\bar{S}_1(k) + Bw(k), A = \Delta^{-1}(F_{11} + F_{12}\Gamma_1), B = \Delta^{-1}(F_{12}\Gamma_2 + \Lambda_1), \\
&\bar{y}(k) = H\bar{S}_1(k) + \bar{v}(k), H = C_1 + C_2\Gamma_1, \bar{v}(k) = C_2\Gamma_2 w(k) + v(k), \\
&E[\bar{v}(k)\bar{v}(s)] = \bar{R}\delta_K(k-s), \bar{R} = C_2\Gamma_2 Q \Gamma_2^T C_2^T + R, \\
&\bar{S}_2(k) = \Gamma_1\bar{S}_1(k) + \Gamma_2 w(k), \Gamma_1 = -F_{22}^{-1}F_{21}, \Gamma_2 = -F_{22}^{-1}\Lambda_2, F_{22} > 0.
\end{aligned} \tag{52}$$

By putting $\bar{S}_1(k)$ as $x(k) = \bar{S}_1(k)$ and A as $\Phi = A$, the state equation for the state vector $x(k)$ and its observation equation, without measurement delays or including uncertain observations, are as follows.

$$\begin{aligned}
&x(k+1) = \Phi x(k) + Bw(k), E[w(k)w(s)] = Q\delta_K(k-s), \\
&\bar{y}(k) = z(k) + \bar{v}(k), z(k) = Hx(k), E[\bar{v}(k)\bar{v}(s)] = \bar{R}\delta_K(k-s)
\end{aligned} \tag{53}$$

Now, let us consider the observation equation in the case of $\bar{N} = 2$ in (6).

$$\begin{aligned}
&y(k) = \bar{\gamma}_1(k)\tilde{y}(k) + \bar{\gamma}_0(k)\tilde{v}(k) = \bar{\gamma}_1(k)\tilde{H}\tilde{x}(k) + \bar{\gamma}_2(k)\tilde{v}(k), \\
&\bar{\gamma}_1(k) = \begin{bmatrix} \gamma_{01}(k) & \gamma_{11}(k) & \gamma_{21}(k) \end{bmatrix}, \bar{\gamma}_0(k) = \begin{bmatrix} \gamma_{00}(k) & \gamma_{10}(k) & \gamma_{20}(k) \end{bmatrix}, \\
&\bar{\gamma}_2(k) = \begin{bmatrix} \gamma_{01}(k) + \gamma_{00}(k) & \gamma_{10}(k) + \gamma_{11}(k) & \gamma_{20}(k) + \gamma_{21}(k) \end{bmatrix}, \\
&\bar{p}_1(k) = \begin{bmatrix} p_{01}(k) & p_{11}(k) & p_{21}(k) \end{bmatrix}, \bar{p}_0(k) = \begin{bmatrix} p_{00}(k) & p_{10}(k) & p_{20}(k) \end{bmatrix}, \\
&\bar{p}_2(k) = \begin{bmatrix} p_{01}(k) + p_{00}(k) & p_{10}(k) + p_{11}(k) & p_{20}(k) + p_{21}(k) \end{bmatrix}, \\
&\tilde{y}(k) = \begin{bmatrix} \bar{y}(k) & \bar{y}(k-1) & \bar{y}(k-2) \end{bmatrix}^T, \\
&\tilde{v}(k) = \begin{bmatrix} \bar{v}(k) & \bar{v}(k-1) & \bar{v}(k-2) \end{bmatrix}^T,
\end{aligned} \tag{54}$$

$$\bar{z}(k)=\begin{bmatrix} z(k) \\ z(k-1) \\ z(k-2) \end{bmatrix}=\begin{bmatrix} Hx(k) \\ Hx(k-1) \\ Hx(k-2) \end{bmatrix}=\bar{H}\bar{x}(k), \bar{x}(k)=\begin{bmatrix} x(k) \\ x(k-1) \\ x(k-2) \end{bmatrix},$$
$$\bar{H}=\begin{bmatrix} H & 0 & 0 \\ 0 & H & 0 \\ 0 & 0 & H \end{bmatrix}=\begin{bmatrix} 0.5 & 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.7 \end{bmatrix}, H=[0.5 \quad 0.7]. \tag{55}$$

The values of the probabilities for the Bernoulli random variables, $\gamma_{ij}(k), i, j=0,1,2,$ satisfy

$$\Pr\{\gamma_{01}(k)\}=\Pr\{\gamma_{01}(k)^2\}=p_{01}, \Pr\{\gamma_{11}(k)\}=\Pr\{\gamma_{11}(k)^2\}=p_{11},$$
$$\Pr\{\gamma_{21}(k)\}=\Pr\{\gamma_{21}(k)^2\}=p_{21}, \Pr\{\gamma_{00}(k)\}=\Pr\{\gamma_{00}(k)^2\}=p_{00},$$
$$\Pr\{\gamma_{10}(k)\}=\Pr\{\gamma_{10}(k)^2\}=p_{10}, \Pr\{\gamma_{20}(k)\}=\Pr\{\gamma_{20}(k)^2\}=p_{20}.$$

The probabilities p_{01}, p_{11}, p_{21}, p_{00}, p_{10} and p_{20} used in the simulation are summarized in Table 1.

The system matrix Φ, which is equal to A, is calculated in (52). Also, from the state equation for $x(k)$ in (53), $K_x(k,k)$ is evaluated as

$$\Phi=\begin{bmatrix} 0 & 0.666667 \\ -0.8 & -0.266667 \end{bmatrix}, K_x(k,k)=\begin{bmatrix} K(0) & K(1) \\ K(1) & K(0) \end{bmatrix},$$
$$K(0)=0.202121, K(1)=0.379147. \tag{56}$$

Table 1 Probabilities for the Bernoulli variables $\gamma_{01}(k)$, $\gamma_{11}(k)$, $\gamma_{21}(k)$, $\gamma_{00}(k)$, $\gamma_{10}(k)$ and $\gamma_{20}(k)$.

Cases of delay	Probability of the observation including both signal and observation noise	Probability of the observation not including signal and consisting of only observation noise
No delay	$\Pr\{\gamma_{01}(k)=1\}=p_{01}(k)=0.8820$	$\Pr\{\gamma_{00}(k)=1\}=p_{00}(k)=0.0180$
One-step delay	$\Pr\{\gamma_{11}(k)=1\}=p_{11}(k)=0.0570$	$\Pr\{\gamma_{10}(k)=1\}=p_{10}(k)=0.0030$
Two-steps delay	$\Pr\{\gamma_{21}(k)=1\}=p_{21}(k)=0.0360$	$\Pr\{\gamma_{20}(k)=1\}=p_{20}(k)=0.0040$

From (16) and (18), $\bar{\Phi}$ and $K_{\bar{x}}(0)$ are given by

$$\bar{\Phi} = \begin{bmatrix} \Phi & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0.666667 & 0 & 0 & 0 & 0 \\ -0.8 & -0.266667 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad (57)$$

$$K_{\bar{x}}(0) = \begin{bmatrix} K_x(0) & \Phi K_x(0) & \Phi^2 K_x(0) \\ K_x(0)\Phi^T & K_x(0) & \Phi K_x(0) \\ K_x(0)(\Phi^T)^2 & K_x(0)\Phi^T & K_x(0) \end{bmatrix}.$$

From (10) $\Phi_{\tilde{V}}$ and $K_{\tilde{V}}(L,L)$ are given by

$$\Phi_{\tilde{V}} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, K_{\tilde{V}}(L,L) = \begin{bmatrix} \bar{R} & 0 & 0 \\ 0 & \bar{R} & 0 \\ 0 & 0 & \bar{R} \end{bmatrix}.$$

$\breve{R}(L)$ is calculated by substituting $\bar{H}$, $K_{\bar{x}}(L,L) = K_{\bar{x}}(0)$, $K_{\tilde{V}}(L,L) = K_{\tilde{V}}(0)$, $\bar{\gamma}_1(L)$, $\bar{\gamma}_2(L)$ with $E[\bar{\gamma}_1(L)] = \bar{p}_1(k)$ and $E[\bar{\gamma}_2(L)] = \bar{p}_2(L)$ into (47). Substituting U, V, Γ_1, $\bar{H}$, $\bar{p}_1(L)$, $\bar{p}_2(L)$, $\bar{\Phi}$, $\Phi_{\tilde{V}}$, $K_{\bar{x}}(k,k)$ and $K_{\tilde{V}}(L,L)$ into the estimation algorithms in Theorem 1, the fixed-point smoothing estimates $\hat{S}_1(k,k+Lag)$ of $S_1(k)$, $\hat{S}_2(k,k+Lag)$ of $S_2(k)$ and $\hat{S}_3(k,k+Lag)$ of $S_3(k)$ are calculated recursively. Here, Lag represents the fixed lag from $k+Lag$ to k. Fig.1 illustrates the fixed-point smoothing estimate $\hat{S}_1(k,k+5)$ vs. k for the white Gaussian observation noise $N(0,0.1^2)$. Fig.2 illustrates the fixed-point smoothing estimate $\hat{S}_2(k,k+5)$ vs. k for the white Gaussian observation noise $N(0,0.1^2)$. Fig.3 illustrates the fixed-point smoothing estimate $\hat{S}_3(k,k+5)$ vs. k for the white Gaussian observation noise $N(0,0.1^2)$. Fig.4 illustrates the mean-square values (MSVs) of the filtering errors $S_1(k) - \hat{S}_1(k,k)$ and the fixed-point smoothing errors $S_1(k) - \hat{S}_1(k,k+Lag)$ vs. Lag, $0 \le Lag \le 10$, for the white Gaussian observation noises $N(0,0.1^2)$, $N(0,0.3^2)$ and $N(0,0.5^2)$. For $Lag = 0$, the MSV of the filtering errors $S_1(k) - \hat{S}_1(k,k)$ is shown. In Fig.4, for each variance of the observation noise, the MSV of the fixed-point smoothing errors is larger than that of the filtering errors. Fig.5 illustrates the MSVs of the filtering errors $S_2(k) - \hat{S}_2(k,k)$ and the fixed-point smoothing errors $S_2(k) - \hat{S}_2(k,k+Lag)$ vs. Lag, $0 \le Lag \le 10$, for the white Gaussian observation noises $N(0,0.1^2)$, $N(0,0.3^2)$ and $N(0,0.5^2)$. Fig.6 illustrates the MSVs of the filtering errors $S_3(k) - \hat{S}_3(k,k)$ and the fixed-point smoothing errors $S_3(k) - \hat{S}_3(k,k+Lag)$ vs. Lag, $0 \le Lag \le 10$, for the white Gaussian observation noises $N(0,0.1^2)$, $N(0,0.3^2)$ and $N(0,0.5^2)$. In Fig.5 and Fig.6, there is a tendency that the MSVs of the fixed-point smoothing errors decrease as Lag increases. Also, in Fig.5 and Fig.6, the estimation accuracy of the fixed-point smoothing estimate is superior to the filtering estimate. Here, the MSVs of the fixed-point smoothing and filtering errors are calculated by $\sum_{k=1}^{2000} (S_i(k) - \hat{S}_i(k,k+Lag))^2 / 2000$ and $\sum_{k=1}^{2000} (S_i(k) - \hat{S}_i(k,k))^2 / 2000$, $i = 1,2,3,$ respectively.

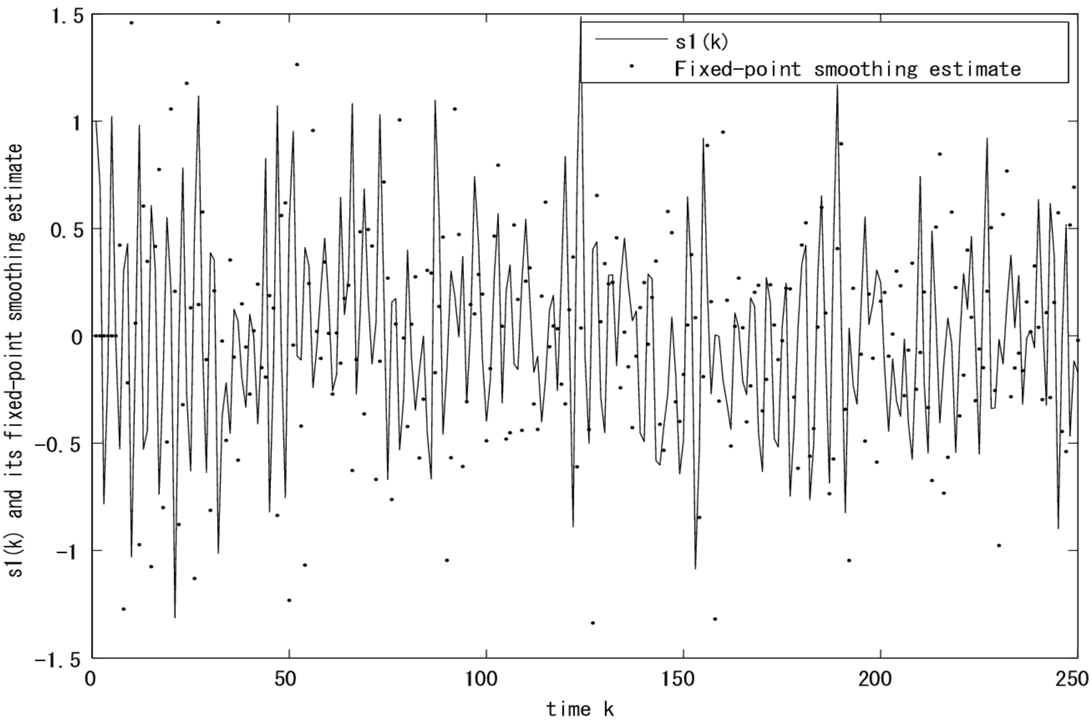

Fig.1 Fixed-point smoothing estimate $\hat{S}_1(k,k+5)$ vs. k for the white Gaussian observation noise $N(0,0.1^2)$.

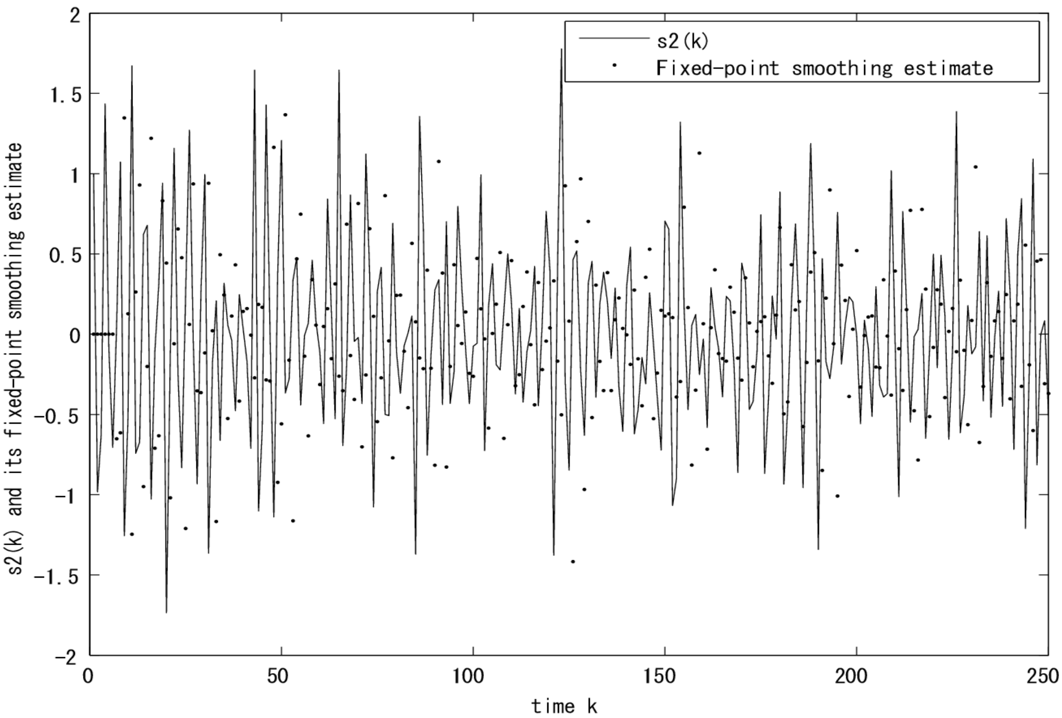

Fig.2 Fixed-point smoothing estimate $\hat{S}_2(k,k+5)$ vs. k for the white Gaussian observation noise $N(0,0.1^2)$.

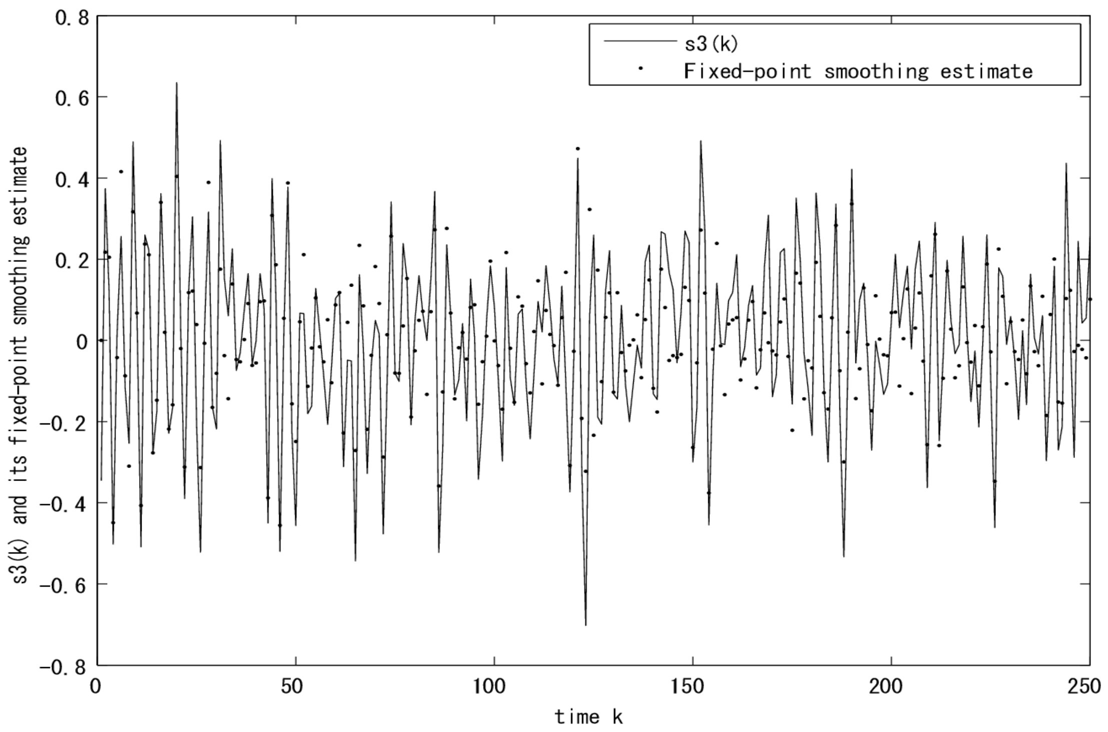

Fig.3 Fixed-point smoothing estimate $\hat{S}_3(k,k+5)$ vs. k for the white Gaussian observation noise $N(0,0.1^2)$.

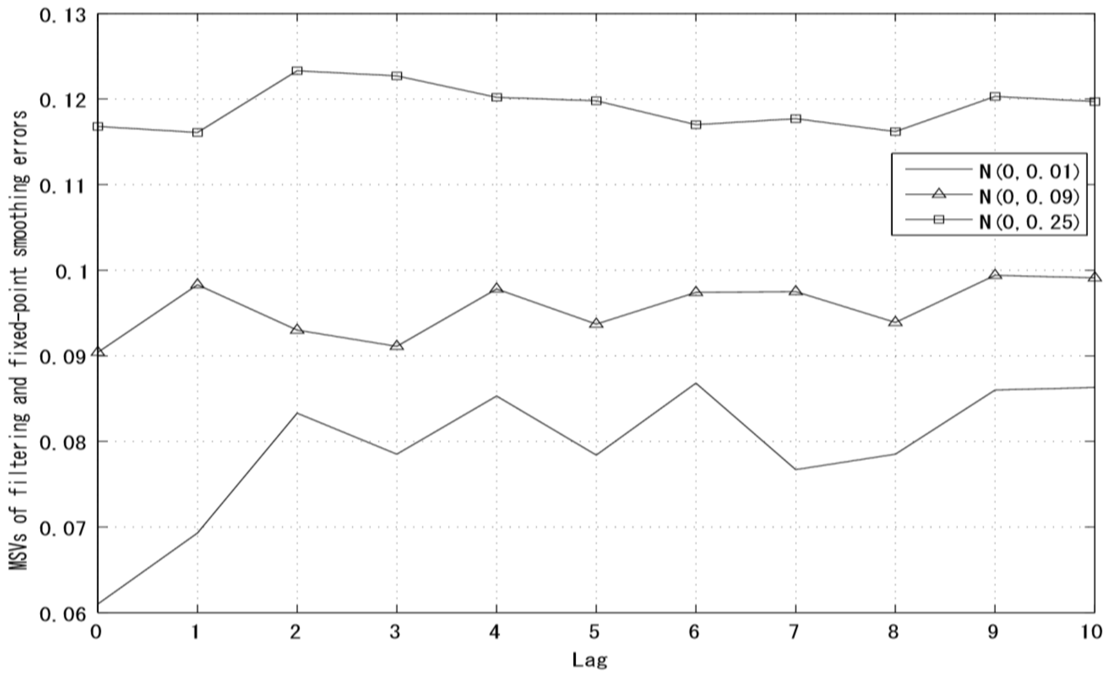

Fig.4 Mean-square values of the filtering errors $S_1(k)-\hat{S}_1(k,k)$ and the fixed-point smoothing errors $S_1(k)-\hat{S}_1(k,k+Lag)$ vs. Lag, $0 \le Lag \le 10$, for the white Gaussian observation noises $N(0,0.1^2)$, $N(0,0.3^2)$ and $N(0,0.5^2)$.

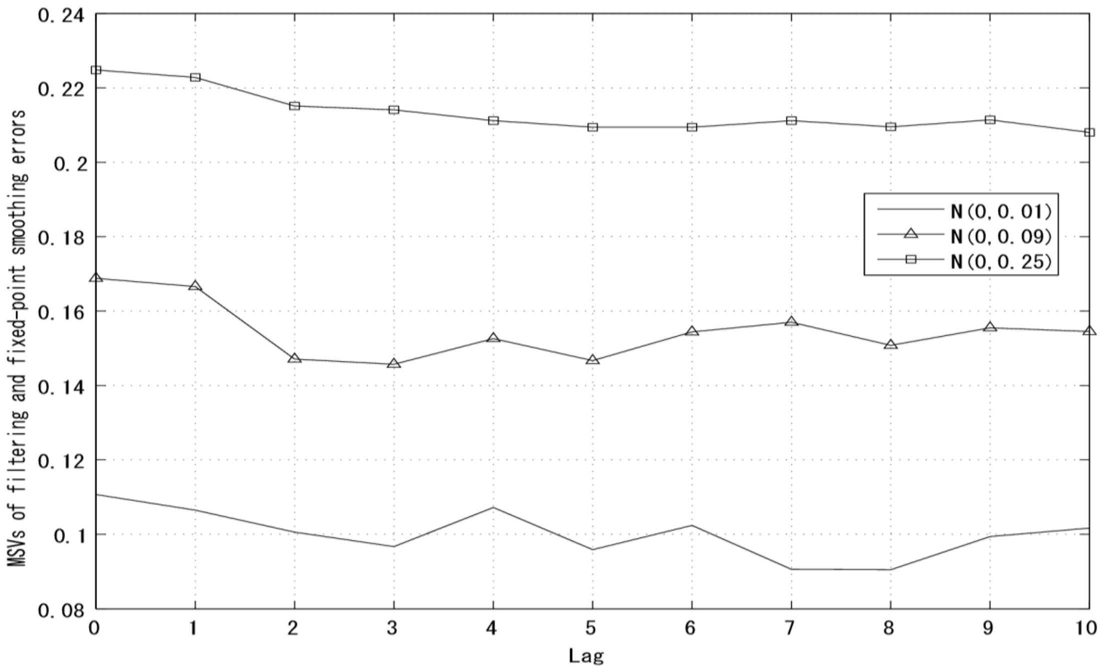

Fig.5 Mean-square values of the filtering errors $S_2(k)-\hat{S}_2(k,k)$ and the fixed-point smoothing errors $S_2(k)-\hat{S}_2(k,k+Lag)$ vs. Lag, $0 \le Lag \le 10$, for the white Gaussian observation noises $N(0,0.1^2)$, $N(0,0.3^2)$ and $N(0,0.5^2)$.

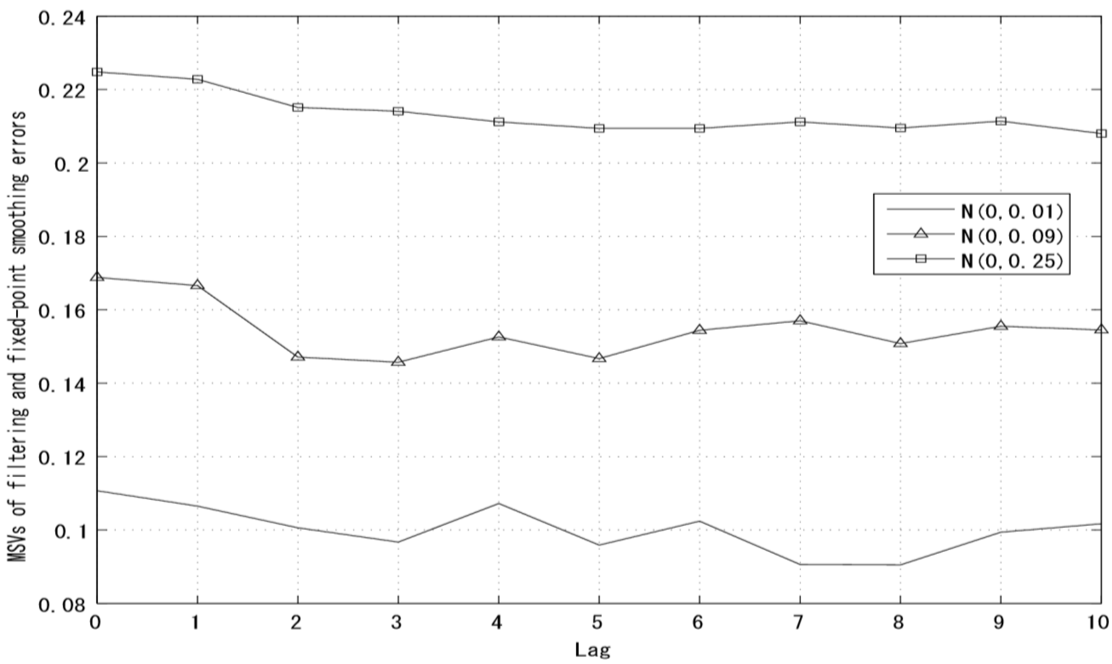

Fig.6 Mean-square values of the filtering errors $S_3(k)-\hat{S}_3(k,k)$ and the fixed-point smoothing errors $S_3(k)-\hat{S}_3(k,k+Lag)$ vs. Lag, $0 \le Lag \le 10$, for the white Gaussian observation noises $N(0,0.1^2)$, $N(0,0.3^2)$ and $N(0,0.5^2)$.

5. Conclusions

In this paper, the RLS Wiener fixed-point smoother and filter are designed for the descriptor systems with randomly delayed, by multiple sampling intervals, or uncertain observations in linear discrete-time stochastic systems. In this paper, in addition to the multiply and randomly delayed observations [12], the uncertain observation in [13] is taken into account particularly for the linear discrete-time descriptor systems. The uncertain observation might correspond to the packet dropout. The packet dropout in the network systems is caused by the nodal delay as the sum of the processing delay, the queuing delay, the transmission delay and the propagation delay. Some numerical simulation results have shown that the devised estimators have feasible estimation characteristics.

Since the RLS Wiener estimators necessitate the information of the variance Q of the input noise and the input matrix Λ in the state equation (1), the estimation accuracy of the proposed RLS Wiener estimators are not degraded by the estimations of Q and Λ.

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